The monster Lie algebra.

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We calculate the multiplicities of all the roots of the "monster Lie algebra". This gives an example of a Lie algebra all of whose simple roots and root multiplicities are known and which is not finite dimensional or an affine Kac-Moody algebra. There seem to be several similar infinite dimensional Lie algebras, which to a sympathetic eye appear to correspond to some of the sporadic simple groups.

1 Introduction

We will prove

Theorem 1. *Let $II_{25,1}$ be the 26 dimensional even unimodular Lorentzian lattice, and let ρ be a Weyl vector for some choice of simple roots of its reflection group W (so ρ has norm 0). We define the monster Lie algebra M to be the generalized Kac-Moody algebra with root lattice $II_{25,1}$ whose simple roots are the simple roots of W, together with the positive multiples of ρ, each with multiplicity 24. Then any nonzero root $r \in II_{25,1}$ of M has multiplicity $p_{24}(1 - r^2/24)$, which is the number of partitions of $1 - r^2/2$ into parts of 24 colours.*

This is equivalent to the following "denominator formula".

$$e^\rho \prod_{r\in\Pi^+} (1 - e^r)^{p_{24}(1-r^2/2)} = \sum_{\substack{w\in W\\ n\in Z}} \det(w)\tau(n)e^{w(n\rho)}$$

where $\tau(n)$ is the Ramanujan tau function, and Π^+ is the set of vectors of $II_{25,1}$ which are either positive multiples of ρ or have negative inner product with ρ.

We give a brief outline of the proof of theorem 1. In section 3 we construct a Lie algebra M from the Lorentzian lattice $II_{25,1}$ using vertex algebras. The "no ghost" theorem ([11]) allows us to calculate the multiplicities of the roots of M, which are as given in theorem 1, and also allows us to prove that M is a generalized Kac-Moody algebra . We can find the real simple roots of M by using Conway's theorem ([8]) which describes the reflection group of the lattice $II_{25,1}$, and we show that the positive multiples of ρ are simple roots of multiplicity 24.

To complete the proof of theorem 1 we have to show that M has no other simple roots. To do this we examine both sides of the "denominator formula" of M. We can evaluate one side of this by using Hecke operators, and it turns out to be a sort of modular form. The other side of the denominator formula is a sum of terms of the form q^n times a modular form, where n is proportional to the norm of $r+\rho$ for some simple root r. By looking at the asymptotic behaviour of both sides at the cusp 0 we prove that all the n's must be 0, which can be used to show that there are no other simple roots and proves theorem 1.

Section 5 contains a number of corollaries of theorem 1; for example any 25 dimensional unimodular positive definite lattice must have a root. In section 6 we give some not very convincing evidence for the existence of some more Lie algebras similar to the monster Lie algebra M associated to some of the other sporadic simple groups.

The results we use here can be found in the following places. The results about vertex algebras we use are taken from Borcherds [1], and proofs of these have recently been written up by Lepowsky, Frenkel and Meurman in their book [15]. The no ghost theorem is proved in Goddard and Thorn [11] and in Frenkel [14]. We use some standard results about Kac-Moody algebras which can all be found in Kac's book [13], and the results about generalized Kac-Moody algebras that we need are in Borcherds [2] and [3] and are summarized in section 2. The results about the geometry of the Leech lattice Λ and the Lorentzian lattice $II_{25,1}$that we need can be found either in the original papers by Conway, Parker, and Sloane, most of which are reprinted in the book [9] by Conway and Sloane, or in [5]. Finally the results on modular forms, Hecke operators and theta functions that we use can be found in any book on modular forms, for example Serre [17] or Gunning [12].

Remark. The multiplicities $p_{24}(1+n)$ of the roots of the monster Lie algebra are given by Rademacher's formula ([16, Chapter 15])

$$p_{24}(1+n) = 2\pi n^{-13/2}\sum_{k>0}\frac{I_{13}(4\pi\sqrt{n}/k)}{k}\sum_{\substack{0\le h,h'<k\\ hh'\equiv -1 \bmod k}} e^{2\pi i(nh+h')/k}$$

where $I_{13}(z)=-iJ_{13}(iz)$ is the modified Bessel function of order 13. In particular $p_{24}(1+n)$ is asymptotic to $2^{-1/2}n^{-27/4}e^{4\pi\sqrt{n}}$ for large n.

Notation.

Λ is the Leech lattice, the unique 24 dimensional even positive definite lattice with no roots. Its elements are denoted by λ.

U is the two dimensional even unimodular Lorentzian lattice, so it is spanned by two norm 0 vectors ρ,ρ' with inner product -1.

$II_{25,1}$ is the even 26 dimensional unimodular Lorentzian lattice. We usually write $II_{25,1}$as $\Lambda\oplus U$, and write elements of $II_{25,1}$in the form $(\lambda,m,n),\lambda\in\Lambda,m,n\in Z$, and this element has norm λ^2-2mn.

ρ is the norm 0 vector $(0,0,1)$ of $II_{25,1}$, so that $\rho^\perp/\rho$ is isomorphic to the Leech lattice Λ.

ρ' is the vector $(0,1,0)$ of $II_{25,1}$.

M is the monster Lie algebra, whose root lattice is $II_{25,1}$.

Π^+ is the set of positive roots of M, which are the vectors $v\in II_{25,1}$ with $v^2\le 2,(v,\rho)<0$ together with the positive multiples of ρ.

Π^S is the collection of simple roots of M counted with multiplicities, which will turn out to be the norm two vectors of $II_{25,1}$of height 1 together with 24 copies of each positive multiple of ρ. The norm 2 vectors of Π^S are the simple roots of the reflection group of $II_{25,1}$.

W is the reflection group of $II_{25,1}$, with typical elements w.

$p_{24}(n)$ is the number of partitions of n into parts of 24 colours, so that $\sum_n p_{24}(1+n)q^n = q^{-1}\prod_{n>0}(1-q^n)^{-24} = \Delta(q)^{-1} = q^{-1}+24+324q+3200q^2+25650q^3+176256q^4+1073720q^5+\ldots$. These are the multiplicities of roots of the monster Lie algebra M.

$\tau(n)$ is the Ramanujan tau function, whose generating function is

$\Delta(q)$ $= \sum_n \tau(n)q^n = q\prod_{n>0}(1-q^n)^{24} = q-24q^2+252q^3+\ldots$.

$\Theta_\Lambda(q)$ $= \sum_{\lambda\in\Lambda} q^{\lambda^2/2} = 1+196560q^2+\ldots$ is the theta function of the Leech lattice.

$c(n)$ are the coefficients of the elliptic modular function $j(q)-720$.

$j(q)$ is the elliptic modular function with $j(q)-720 = \sum_n c(n)q^n = q^{-1}+24+196884q+\ldots = \Theta_\Lambda(q)/\Delta(q)$

q, τ are complex variables, with $q = e^{2\pi i\tau}$, $|q|<1$, and $\mathrm{Im}(\tau)>0$.

2 Generalized Kac-Moody algebras

We recall the results about generalized Kac-Moody algebras that we need. We will modify the original definition of generalized Kac-Moody algebras in [2] slightly so that these algebras are closed under taking universal central extensions, as in [3].

Suppose that $a_{ij}, i, j \in I$ is a symmetric (possibly infinite) real matrix such that $a_{ij} \le 0$ if $i \ne j$ and such that if $a_{ii} > 0$ then $2a_{ij}/a_{ii}$ is an integer for any j. Then the universal generalized Kac-Moody algebra of this matrix is defined to be the Lie algebra generated by elements e_i, f_i, h_{ij} for $i, j \in I$ satisfying the following relations:

1. $[e_i, f_j] = h_{ij}$
2. $[h_{ij}, e_k] = \delta_i^j a_{ik} e_k, [h_{ij}, f_k] = -\delta_i^j a_{ik} f_k$
3. If $a_{ii} > 0$ then $\mathrm{Ad}(e_i)^n e_j = \mathrm{Ad}(f_i)^n f_j = 0$, where $n = 1 - 2a_{ij}/a_{ii}$.
4. If $a_{ii} \le 0, a_{jj} \le 0$ and $a_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$.

If $a_{ii} > 0$ for all $i \in I$ then this is the same as the Kac-Moody algebra with symmetrized Cartan matrix a_{ij}. In general these algebras have almost all the good properties that Kac-Moody algebras have, and the only major difference is that generalized Kac-Moody algebras are allowed to have imaginary simple roots. We list some of their properties from [2] and [3].

(1) The element h_{ij} is 0 unless the i'th and j'th columns of a are equal. The elements h_{ij} for which the i'th and j'th columns of a are equal form a basis for an abelian subalgebra H of G, called its Cartan subalgebra. (In the case of Kac-Moody algebras, the i'th and j'th columns of a cannot be equal unless $i = j$, so the only nonzero elements h_{ij} are those of the form h_{ii}, which are usually denoted by h_i.) Every nonzero ideal of G has nonzero intersection with H. The centre of G is contained in H and contains all the elements h_{ij} for $i \ne j$. If a is not the direct sum of two smaller matrices, not the 1×1 zero matrix, and not the matrix of an affine Kac-Moody algebra, then G modulo its centre is simple.

(2) If a has no zero columns then G is perfect and equal to its own universal central extension. (This is why we need the elements h_{ij} for $i \ne j$.)

(3) Suppose we choose a positive integer n_i for each $i \in I$. Then we can grade G by putting $\deg(e_i) = -\deg(f_i) = n_i$. The degree 0 piece of G is the Cartan subalgebra H.

(4) G has an involution ω with $\omega(e_i) = -f_i, \omega(f_i) = -e_i$, called the Cartan involution. There is an invariant inner product $(,)$ on G such that $(e_i, f_i) = 1$ for all i, and it also has the property that $-(g, \omega(g)) > 0$ whenever g is a homogeneous element of nonzero degree.

(5) There is a character formula for simple highest weight modules M_λ of G with highest weight λ, which states that

$$Ch(M_\lambda)e^\rho \prod_{\alpha\in\Pi^+}(1-e^\alpha)^{\mathrm{mult}(\alpha)} = \sum_{w\in W}\det(w)w(e^\rho\sum_\alpha \epsilon(\alpha)e^{\alpha+\lambda})$$

The only case of this we need is for the one dimensional module, when it becomes the denominator formula

$$e^\rho \prod_{\alpha\in\Pi^+}(1-e^\alpha)^{\mathrm{mult}(\alpha)} = \sum_{w\in W}\det(w)w(e^\rho\sum_\alpha \epsilon(\alpha)e^{\alpha})$$

Here ρ is the Weyl vector (i.e. a vector with $(\rho, r) = -r^2/2$ for all simple roots r), Π is the set of positive roots α, W is the Weyl group, and $\epsilon(\alpha)$ is $(-1)^n$ if α is the sum of n distinct pairwise perpendicular imaginary simple roots that are all perpendicular to λ, and 0 otherwise. This formula is still correct if ρ is replaced by a vector which has inner product $-r^2/2$ with all *real* simple roots r, because $e^{w(\alpha+\rho)-\rho}$ depends only on the inner product of ρ with the real simple roots. If G is a Kac-Moody algebra then there are no imaginary simple roots so the sum over α is 1 and we recover the usual character and denominator formulas.

We also need the following characterization of generalized Kac-Moody algebras from [2].

Theorem 2·. *A Z-graded Lie algebra $G = \bigoplus_{i\in Z} G_i$ is the quotient of a generalized Kac-Moody algebra graded as in (3) above by a subspace of its centre if and only if it has the following four properties.*

1 $G_0 \subset [G, G]$
2 *G has an involution ω which acts as -1 on G_0 and maps G_i to G_{-i}*
3 *G has an invariant bilinear form $(,)$ such that G_i and G_j are orthogonal if $i \neq -j$, and such that $-(g, \omega(g)) > 0$ if g is a nonzero homogeneous element of G of nonzero degree.*
4 *As a module over G_0, G is a sum of finite dimensional submodules.*

(If condition (1) is omitted it is still easy to describe G, by adding some outer derivations to a generalized Kac-Moody algebra .)

3 The monster Lie algebra.

We construct a generalized Kac-Moody algebra algebra M, called the monster Lie algebra, with root lattice $II_{25,1}$whose root multiplicities are given by the number of partitions of an integer into parts of 24 colours. We find the real simple roots of this algebra by using Conway's description of the reflection group of $II_{25,1}$, and we show that all positive multiples of ρ are simple roots of multiplicity 24.

In this section we will prove

Theorem 3. *There is a generalized Kac-Moody algebra M with root lattice $II_{25,1}$such that the multiplicity of the nonzero vector $r \in II_{25,1}$ is $p_{24}(1 - r^2/2)$. The real simple roots of M are the norm 2 vectors r with (r, ρ) = -1, and the positive multiples of ρ are simple roots of multiplicity 24.*

We will later show that M has no other simple roots.

Lemma 1. *There is a Lie algebra M with the following properties.*

1. *M is graded by $II_{25,1}$. The piece of degree $r \in II_{25,1}, r \neq 0$ has dimension $p_{24}(1 - r^2/2)$.*
2. *M has an involution ω, which acts as -1 on $II_{25,1}$ and on the piece of M of degree $0 \in II_{25,1}$.*
3. *M has a contravariant bilinear form (,) such that the pieces of M of degrees $i, j \in II_{25,1}$ are orthogonal if $i \neq j$ and such that (,) is positive definite on the piece of M of degree $i \in II_{25,1}$ if $i \neq 0$.*

Proof: This Lie algebra was constructed in Borcherds [1]; we briefly recall its construction. Given any nonsingular even lattice L we can can construct its vertex algebra V, which is an L-graded vector space with a large number of operations. In particular there is an action of the Virasoro algebra on it, where the Virasoro algebra is spanned by the operators 1 and $L_i, i \in Z$ with the relations

$$[L_i, L_j] = (i - j)L_{i+j} + \frac{i^3 - i}{12} \dim(L)\delta^i_{-j}$$

We let P^i be the space of vectors $v \in V$ satisfying $L_0(v) = iv$, $L_n(v) = 0$ if $n < 0$. Then one of the results of [1] is that $P^1/L_1(P^0)$ can be made into a Lie algebra. The space V has an involution ω with the property (2) above and an inner product (,) which when restricted to P^1 is singular on $L_1(P^0)$ and defines a contravariant inner product on the Lie algebra $P^1/L_1(P_0)$. The pieces of this Lie algebra of degrees $i, j \in L$ are orthogonal unless $i = j$.

So far this construction can be done for any even lattice. In the special case of a 26 dimensional Lorentzian lattice, the "no ghost" theorem of Goddard and Thorn [11] implies that if $r \in L, r \neq 0$ then the restriction of the inner product (,) to the degree r piece of $P^1/L_1(P^0)$ is positive semidefinite and the quotient by its kernel has dimension $p_{24}(1 - r^2/2)$. The fact that (,) is contravariant implies that its kernel is an ideal of $P^1/L_1(P^0)$, so if we define M to be the quotient of $P^1/L_1(P^0)$ by the kernel of (,), for $L = II_{25,1}$, then M has all the properties stated in the lemma. Q.E.D.

Lemma 2. *The Lie algebra M is a generalized Kac-Moody algebra .*

Proof. If we fix any negative norm vector r of $II_{25,1}$not perpendicular to any norm 2 vectors, then we can make M into a Z-graded Lie algebra by using the inner product with r as the degree. All the conditions of theorem 2 are satisfied for M, so M is a generalized Kac-Moody algebra .

We now fix a primitive norm 0 vector ρ of $II_{25,1}$which is not perpendicular to any norm 2 vectors. (All such vectors ρ are conjugate under $\mathrm{Aut}(II_{25,1})$.) We choose a fundamental domain for the reflection group W of $II_{25,1}$containing ρ.

Lemma 3. *The positive norm simple roots of M are the norm 2 vectors $r \in II_{25,1}$ with $(r,\rho) = -1$.*

Proof: All norm 2 vectors of $II_{25,1}$have multiplicity 1, so the positive norm simple roots of M are just the simple roots of the reflection group of $II_{25,1}$. This reflection group was described by Conway [8] who showed that its simple roots are the norm 2 vectors r of $II_{25,1}$with $(r,\rho) = -1$. Q.E.D.

Lemma 4. *The positive multiples of ρ are simple roots of multiplicity 24.*

Proof. As $(\rho,\rho) = 0$ and $(\rho, r) < 0$ for all other simple roots r, no multiple of ρ can be written as a sum of simple roots, unless all the roots in the sum are perpendicular to each other. Hence the simple root $n\omega, n > 0$ has multiplicity equal to the multiplicity of the root $n\omega$, which is 24. Q.E.D.

4 There are no other simple roots.

We complete the proof of theorem 1 by showing that the Lie algebra M of the previous section has no other simple roots.

Lemma 1.

$$e^\rho \prod_{r\in\Pi^+} (1-e^r)^{p_{24}(1-r^2/2)} = \sum_{w\in W} \det(w) w(\sum_{n\in Z} \tau(n) e^{n\rho} - \sum_{r\in\Pi^S, r^2<0} e^{r+\rho})$$

where Π^+ is the set of positive roots of the monster Lie algebra, and Π^S is the collection of simple roots of the monster Lie algebra, counted with multiplicities.

Proof. We show that this is just the denominator formula of section 2 for the generalized Kac-Moody algebra M. The vector ρ has inner product $-1 = r^2/2$ with all real simple roots r by Conway's theorem (Lemma 3 above), so to prove this is the denominator formula we have to check that

$$\sum_{n\in Z} \tau(n) e^{n\rho} - \sum_{r\in\Pi^S, r^2<0} e^{r+\rho} = e^\rho \sum_\alpha \epsilon(\alpha) e^\alpha$$

where $\epsilon(\alpha)$ is the sum of a term $(-1)^n$ for each way of writing α as a sum of n distinct pairwise perpendicular imaginary simple roots. Two imaginary roots can only be perpendicular if they are both of norm 0 and proportional, so if α is not a multiple of ρ then $\epsilon(\alpha)$ is just the multiplicity of α as a simple root. All positive multiples of ρ have multiplicity 24 and are perpendicular to each other, so $\epsilon(n\rho)$ is the coefficient of q^n in $\prod_n (1-q^n)^{24} = q^{-1}\sum_n \tau(n) q^n$, which proves lemma 1.

We recall that $II_{25,1} = \Lambda \oplus U$, where U is a two dimensional lattice spanned by ρ, ρ' with $\rho^2 = \rho'^2 = 0, (\rho, \rho') = -1$. We project both sides of the denominator formula from the group ring of the lattice $II_{25,1}$onto the group ring of the lattice U, which makes them easier to calculate. More precisely, we define the projection P from the group ring of $II_{25,1}$to the space of Laurent series in p and q by

$$P(e^r) = p^{-(\rho,r)} q^{-(\rho',r)}$$

so that P applied to each side of the denominator formula is a Laurent series in p and q.

Lemma 2. *If $m > 0$ then*

$$-\log(P(\prod_{r\in\Pi^+}(1-e^r)^{p_{24}(1-r^2/2)})) = \sum_{m>0} T_m(j(q)-720)p^m + 24\sum_{n>0}\frac{\sigma(n)}{n}q^n$$

where $\sigma(n)$ is the sum of the divisors of n, $j(q)$ is the elliptic modular function, and T_m is a Hecke operator (see Serre [17], especially proposition 12 of chapter VII, section 5). In particular the coefficient of p^m for $m > 0$ is a modular function of q of level 1.

Proof. The number of vectors of norm $2a$ of $II_{25,1}$projecting onto the vector $n\rho+m\rho'$ of U is the number of vectors of Λ of norm $2a+2mn$, so the sum of the multiplicities of the roots of M that project onto $n\rho+m\rho' \in U$ is equal to the coefficient $c(mn)$ of $\Theta_\Lambda(q)\sum_n p_{24}(1+n)q^n = j(q)-720 = \sum_n c(n)q^n$. The left hand side of the formula above is therefore equal to

$$\begin{aligned}
&-\log\prod_{m\geq 0}\prod_{n\in Z, n>0 \text{if } m=0}(1-p^m q^n)^{c(mn)} \\
&= \sum_{m\geq 0}\sum_{n\in Z,n>0 \text{ if } m=0}\sum_{k\geq 1} c(mn)p^{mk}q^{nk}/k \\
&= \sum_{m>0}\sum_{n\in Z}\sum_{0<k|(m,n)}\frac{1}{k}c(\frac{mn}{k^2})q^n p^m + \sum_{n>0}\sum_{0<k|n}\frac{1}{k}c(0)q^n \\
&= \sum_{m>0} T_m(\sum_{n\in Z} c(n)q^n)p^m + c(0)\sum_{n>0}\frac{\sigma(n)}{n}q^n
\end{aligned}$$

which is equal to the right hand side. Q.E.D.

Lemma 3.

$$P(e^\rho\prod_{r\in\Pi^+}(1-e^r)^{p_{24}(1-r^2/2)}) = \Delta(q)\Theta_\Lambda(p)-\Theta_\Lambda(q)\Delta(p)$$

where

$$\Delta(q) = q\prod_{n>0}(1-q^n)^{24} = \sum_n \tau(n)q^n = q-24q^2+252q^3\ldots$$

is the generating function of Ramanujan's tau function, and

$$\Theta_\Lambda(q) = \sum_{\lambda\in\Lambda} q^{\lambda^2/2} = 1+196560q^2+\ldots$$

is the theta function of the Leech lattice Λ. In particular the coefficient of p^m is a modular form in q of weight 12 which is holomorphic at all cusps.

Proof. The left hand side is equal to

$$q\exp(\log(P(\prod_{r\in\Pi^+}(1-e^r)^{p_{24}(1-r^2/2)}))) = q\prod_i(1-q^i)^{24}\exp(-\sum_{m>0}p^m T_m(j(q)-720))$$

by lemma 2, so the coefficient of p^m is a modular form of weight 12 and level 1 for any m, because $q\prod_i(1-q^i)^{24}$ is a modular form of weight 12 and level 1, and each of the functions $T_m(j(q)-720)$ is a modular function of level 1.

Therefore if we let $F(p,q)$ stand for the left hand side of the expression above, it has the properties that the coefficient of p^m is a modular form of weight 12 and level 1 for each m and is zero for $m<0$. Moreover $F(p,q)=-F(q,p)$ because it is antisymmetric under reflection in the root $\rho-\rho'$ of $II_{25,1}$. We will show that there is only one function $F(p,q)$ with these properties whose coefficient of q^1p^0 is 1, which will prove lemma 3 because $\Delta(q)\Theta_\Lambda(p)-\Theta_\Lambda(q)\Delta(p)$ also has these properties.

The coefficients must all be modular forms holomorphic at the cusp ∞ because the coefficient of q^np^m is 0 if $n<0$ by antisymmetry. We will now repeatedly use the fact that a holomorphic modular form of weight 12 and level 1 is determined by its coefficients of q^0 and q^1. The coefficient of p^0q^0 of F is 0 by antisymmetry and the coefficient of p^0q^1 is 1 by assumption, so the coefficient of p^0q^n is determined for all n because the coefficient of p^0 is a modular form of weight 12 and level 1. Similarly the coefficients of p^1q^0 and p^1q^1 are -1 and 0 by antisymmetry, so the coefficient of p^1q^n is determined. Finally for any m the coefficients of p^mq^0 and p^mq^1 are determined by antisymmetry, so the coefficient of p^mq^n is determined. This proves lemma 3.

This identifies the left hand side of the denominator formula. We now examine the right hand side by splitting up the terms into orbits under an action of the Leech lattice.

We define the height $\mathrm{ht}(r)$ of a vector $r\in II_{25,1}$ to be $-(r,\rho)$. We prove that there are no simple roots of negative norm by induction on the height, so we assume that we are given a positive integer m such that there are no simple roots of negative norm of height less than m, and we will prove there are none of height m.

Lemma 4.
$$p^{-m}P(\sum_{\substack{w\in W,n\in Z\\ \mathrm{ht}(nw(\rho))=m}}\det(w)\tau(n)e^{nw(\rho)}-\sum_{r\in\Pi^S,\mathrm{ht}(r)=m}e^{\rho+r})$$
is a modular form of weight 12 and level 1.

Proof. If r is any simple root of negative norm, then $\mathrm{ht}(w(r))>\mathrm{ht}(r)$ for any nontrivial element w of W. By assumption there are no simple roots of negative norm of height less than m, so the only way that $w(e^{r+\rho})$ can have height m for some imaginary simple root r is that either r is a multiple of ρ or $w=1$. Therefore the sum in lemma 4 is the coefficient of p^m of
$$\sum_{w\in W}\det(w)w(\sum_{n\in Z}\tau(n)e^{n\rho}-\sum_{r\in\Pi^S,r^2<0}e^{r+\rho})$$
and lemma 4 now follows from lemmas 1 and 3.

Lemma 5. *Recall that $II_{25,1}=\Lambda\oplus U$. We write vectors of $II_{25,1}$in the form (v,m,n) with $v\in\Lambda$, $m\in Z$ and $n\in Z$, and this vector has norm λ^2-2mn. For each $\lambda\in\Lambda$ the map taking $(v,m,n)\in II_{25,1}$ to $(v+m\lambda,m,n+(v,\lambda)+m\lambda^2/2)$ is an automorphism of $II_{25,1}$, and this defines an action of Λ on $II_{25,1}$.*

Proof: An easy check. (This gives a natural identification of $\rho^\perp/\rho=\Lambda$ with the group of automorphisms of $II_{25,1}$which fix ρ and every vector of $\rho^\perp/\rho$.)

Lemma 6. *For any nonzero integer m there are only a finite number of orbits of vectors of $II_{25,1}$of some fixed norm and height m under the action of Λ.*

Proof: Any vector $(v, m, n) \in II_{25,1}$ of given norm and nonzero height is determined by v, so by lemma 5 the number of orbits of such vectors under the action of Λ is at most the order m^{24} of $m\Lambda/\Lambda$. Q.E.D.

Lemma 7. *Suppose that A is any orbit of vectors of $II_{25,1}$of norm $-2n$ and height m under the action of Λ. Then*

$$p^{-m}P(\sum_{a\in A} e^a) = q^{n/m}\Theta_A(q)$$

where $\Theta_A(q)$ is a modular form of weight 12 with nonnegative coefficients and "constant term" m^{-12} at the cusp 0. (The constant term of a modular form $\Theta(q)$ of weight k at the cusp 0 is the limit of $\tau^k\Theta(e^{2\pi i\tau})$ as τ tends to 0 along the positive imaginary axis.)

Proof. Let $(v, m, (v^2/2 + n)/m)$ be any vector in the orbit A. Then

$$\begin{aligned}
p^{-m}P(\sum_{a\in A} e^a) &= \sum_{a\in A} q^{-(a,\rho')} \\
&= \sum_{\lambda\in\Lambda} q^{-((v+m\lambda,m,(v^2/2+n)/m+(v,\lambda)+m\lambda^2/2),\rho')} \\
&= \sum_{\lambda\in\Lambda} q^{v^2/2m+n/m+(v,\lambda)+m\lambda^2/2} \\
&= q^{n/m}\sum_{\lambda\in\Lambda} q^{m(\lambda-v/m)^2/2} \\
&= q^{n/m}\Theta_A(q)
\end{aligned}$$

where $\Theta_A(q)$ is the theta function of a coset of the lattice $\sqrt{m}\Lambda$ of determinant m^{24} and is therefore a modular form of weight $\dim(\Lambda)/2 = 12$ with nonnegative coefficients. The generalized Jacobi inversion formula (Gunning [12], section 20) states that

$$\Theta_A(e^{2\pi i\tau}) = (-i\tau)^{-12}\det(\sqrt{m}\Lambda)^{-1/2}\sum_{\lambda\in\Lambda} e^{2\pi i((v,\lambda)-\lambda^2/2\tau)/m}$$

so $\Theta_A(q)$ has constant term $(\det(\sqrt{m}\Lambda))^{-1/2} = m^{-12}$ at the cusp 0. Q.E.D.

Lemma 8. *The sum*

$$\sum_A q^{n(A)/m}\Theta_A(q)$$

where the sum is over all orbits A of simple roots of height m, is a modular form of weight 12. Each $n(A)$ is a positive integer equal to $-(a+\rho)^2/2 = m - a^2/2$ for $a \in A$ and there are only a finite number of orbits A with any given value of $n(A)$. Each function $\Theta_A(q)$ is *a modular form of weight 12 with nonnegative coefficients which has constant term m^{-12} at the cusp 0.*

Proof. This follows from lemmas 4, 6, and 7 because if a is a vector of norm at most 0 with $(a, \rho) \leq 0$ which is not a multiple of ρ then $(a+\rho)^2 < 0$.

We now complete the proof of theorem 1 by showing that the sum in lemma 8 must be empty. We will do this by studying the behaviour of the sum as τ tends to 0 through values on the positive imaginary axis.

The sum in lemma 8 is a sum of power series with positive coefficients whose formal sum converges for $|q| < 1$, so the sum of the power series also converges for $|q| < 1$. If $f(\tau)$ is a modular form of weight 12 with constant term c at the cusp 0 then

$$f(\tau) = \tau^{-12}c + s(\tau),$$

where $s(\tau)$ tends to 0 faster than any power of τ as τ tends to 0 along the imaginary axis. Therefore

$$\tau^{-12}c + s(\tau) = \sum_A m^{-12}(e^{2\pi i n(A)\tau/m}\tau^{-12} + s_A(\tau))$$

where each term in the sum on the right is nonnegative, and s and s_A are functions tending to zero faster than any power of τ as τ tends to zero along the imaginary axis. In particular the right hand side is bounded as τ tends to 0, so there are only a finite number of terms in the sum because each term on the right tends to m^{-12}. Therefore

$$c - \sum_{n>0} m^{-12} r(n) e^{2\pi i n\tau/m}$$

tends to 0 faster than any power of τ as τ tends to 0 along the positive imaginary axis, where $r(n)$ is the number of orbits A with $n(A) = n$. All the $r(n)$'s are nonnegative integers, so this is impossible unless they are all 0, so there are no simple roots of norm at most 0 and height m. This proves theorem 1.

5 Corollaries

We deduce several results about the monster Lie algebra and about lattices from our main result.

Corollary 1. *The universal central extension $\hat{M}$ of the monster Lie algebra M is a $II_{25,1}$-graded Lie algebra, and if $0 \neq r \in II_{25,1}$ then the subspace of $\hat{M}$ of degree r is mapped isomorphically onto that of M. The subspace of $\hat{M}$ of degree $0 \in II_{25,1}$ (i.e. its "Cartan subalgebra") can be represented naturally as the sum of a one dimensional space for for each vector of the Leech lattice and a space of dimension $24^2 = 576$ for each positive integer.*

Proof. This follows from the description of the Cartan subalgebra of the universal central extension of a generalized Kac-Moody algebra in [3] or section 2 as the sum of a space of dimension n^2 for each simple root of multiplicity n. The monster Lie algebra M has a real simple root of multiplicity 1 for each vector of the Leech lattice, and a simple root of multiplicity 24 for each positive multiple of ρ. Q.E.D.

Corollary 2.

$$e^{\rho}\prod_{r\in\Pi^{+}}(1-e^{r})^{p_{24}(1-r^{2}/2)}=\sum_{w\in W}\sum_{n\in Z}\det(w)\tau(n)e^{\omega(n\rho)}$$

This is just the denominator formula for the generalized Kac-Moody algebra M.

Corollary 3. *If $r\in\Pi^{+}$ then*

$$(r+\rho)^{2}m(r)=\sum_{\substack{\alpha,\beta\in\Pi^{+}\\ \alpha+\beta=r}}(\alpha,\beta)m(\alpha)m(\beta)$$

where $m(r)$ is defined to be $\sum_{n>0,n\in Z,r/n\in II_{25,1}}\mathrm{mult}(r/n)/n$ (which is equal to the multiplicity of r if r is primitive).

This is just the Peterson recursion formula for the multiplicities of the roots of M, which is valid for generalized Kac-Moody algebras . Notice that for the Peterson recursion formula to hold ρ must satisfy $(\rho,r)=-r^{2}/2$ for all simple roots r (not just the real ones), which would not be true if M had any simple roots r of negative norm.

Corollary 4. *Let N be the subalgebra of the monster Lie algebra M generated by the elements of M whose degree has norm 2 (so that N is the Kac-Moody subalgebra of M generated by the real simple roots of N). Then the multiplicity of the root $r\in II_{25,1}$ of N is at most $p_{24}(1-r^{2}/2)$, and equality holds whenever r is in the fundamental domain of W and $\mathrm{ht}(r)>|r^{2}|$.*

Proof. The inequality for the multiplicity of r is clear because N is a subalgebra of M. If r is in the fundamental domain of W, then r has inner product at most 0 with all simple roots, so it is not possible for r to be the sum of ρ and some other simple roots if $|(r,\rho)|=\mathrm{ht}(r)>r^{2}$.

Remark. The algebra N is the original "monster Lie algebra" defined by Conway, Queen and Sloane in [7]. (This definition missed out the norm 0 simple roots of M, possibly because algebras with imaginary simple roots had not been studied at the time.)

Remark. The inequality of corollary 4 for the multiplicities of the roots of N was first proved by Frenkel [14] by showing that the space $P^{1}/L_{1}P^{0}$ of section 3 is a module for N. The lattice $II_{25,1}$has 121 orbits of vectors of norm -2, all but 2 of which have multiplicity 324 as roots of N, and has 665 orbits of vectors of norm -4, all but 3 of which have norm 3200 as roots of N.

Recall that in lemma 1 of section 3 we defined some subspaces P^{i} of the space V consisting of lowest weight vectors of the Virasoro algebra. The elements of the monster Lie algebra commute with all elements of the Virasoro algebra, considered as endomorphisms of the space V of section 3, so all the spaces P^{i} are representations of the monster Lie algebra.

Corollary 5. *If $i<0$ then the space P^{i} is a sum of highest weight and lowest weight representations of M.*

Proof. The height of any weight of $P^{i},i<0$ cannot be 0 as all its weights have negative norm. As M is generated by elements of height -1,0 or 1 this means that P^{i}_{+} and

P^i_- are both representations of M, where P^i_+ and P^i_- are the sums of the elements of P^i whose degrees have positive or negative height. The representation P^i_+ has the property that the norms of its weights are bounded above (by $2i$) and its weights all have negative inner product with all imaginary simple roots of M, so by the results on representations of generalized Kac-Moody algebras in [2] P^i_+ is a sum of lowest weight representations of M. Similarly P^i_- is a sum of highest weight representations.

Corollary 6. *Suppose that n is 1 or a prime. Then the height of vectors v of norm $-2n$ in the fundamental domain of the reflection group of $II_{25,1}$is a linear function of the theta function of the lattice $v^\perp$.*

Proof. The Peterson recursion formula shows that for primitive vectors of any fixed negative norm the height is a linear function of the theta functions of the cosets $a+v^\perp$ for a in the dual of the lattice $v^\perp$. If n is 1 or a prime then the theta function of any of these cosets is a linear function of the theta function of the dual of $v^\perp$, which is in turn a linear function of the theta function of $v^\perp$.

Corollary 7. *Let v be a vector of $II_{25,1}$of negative norm, and suppose for simplicity that there is no norm 0 vector z such that $|(z,v)| < |(v,v)/2|$. Let n be the number of norm 0 vectors z with $(v,z) = (v,v)/2$.*

1. *If v has norm -2 then its height is $(r + 18 - 4n)/12$.*
2. *If v has norm -4 then its height is $(r + 20 - 2n)/8$.*
3. *If v has norm -6 then its height is $(r + 20 - n)/6$.*

Proof. These are just special cases of corollary 6.

Remark. If v has inner product -1 with some norm 0 vector of height h then $v^\perp$ is isomorphic to the sum of a Niemeier lattice and a 1 dimensional lattice, and v has height $2h + 1$.

Remark. The results of corollary 7 for vectors of norm -2 and -4 were originally proved by different methods in [4]. There is a list of the 121 orbits of norm -2 vectors and the 665 orbits of norm -4 vectors of $II_{25,1}$in [4].

Corollary 8.

1. *If L is an even 25 dimensional positive definite lattice of determinant 2 then the number of roots of L is 6 mod 12 unless L is the sum of a Niemeier lattice and a 1 dimensional lattice.*
2. *If L is a 25 dimensional unimodular positive definite lattice which is not the sum of a Niemeier lattice and a one dimensional lattice then the number of norm 2 vectors plus twice the number of norm 1 vectors is 4 mod 8 (and in particular any such lattice has roots).*
3. *If L is a 26 dimensional even, positive definite lattice of determinant 3 with no roots of norm 6 then the number of roots of L is 0 mod 6.*

Proof. This follows from corollary 7 by using the following facts. There is a 1:1 correspondence between orbits of norm -2 vectors of $II_{25,1}$and even 25 dimensional positive definite lattices of determinant 2, such that $v^\perp$ is isomorphic to L. There is a 1:1 correspondence between orbits of norm -4 vectors of $II_{25,1}$and 25 dimensional positive definite unimodular lattices L such that $v^\perp$ is isomorphic to the sublattice of even vectors of L, and the number of norm 1 vectors of L is equal to the number of norm 0 vectors which have inner product -2 with v. Finally there is a 1:1 correspondence between norm -6 vectors v of $II_{25,1}$and roots r of 26 dimensional even positive definite lattices of determinant 3 such that $v^\perp$ is isomorphic to $r^\perp$. In each case we can work out the theta function of L from that of $v^\perp$, and this gives the results of corollary 8.

Remark. In [4] there is an algorithm for classifying the even 26 dimensional positive definite lattices of determinant 3. In particular there is a unique such lattice with no roots, and its automorphism group is $2 \times {}^3D_4(2)$. There are also unique such lattices with root systems a_1^3, a_2, and g_2, so the congruence above is best possible.

Finally we have the following completely useless curiosity. (There are presumably even more curious and useless such formulae for vectors of norms -4, -6,... .)

Corollary 9. *Suppose v is a norm -2 vector of $II_{25,1}$not of height 1 that is in the fundamental chamber of the reflection group of $II_{25,1}$. Then*

$$d(d-51)/2 - \text{number of components of the Dynkin diagram of } v^\perp$$
$$+ \text{number of orbits of } R \text{ under } \sigma = 324.$$

Here d is the rank of the Dynkin diagram of $v^\perp$, R is the set of simple roots of the reflection group of $II_{25,1}$which have inner product -1 with v, and σ is minus the opposition involution of $v^\perp$, which acts on the set R.

Proof. In [4] it is shown that the left hand side of this is the multiplicity of the root v of M, so corollary 9 follows from theorem 1.

6 Lie algebras for other simple groups.

There appear to be many other non affine Kac-Moody algebras similar to the one in this paper, many of which seem to be associated in a rather obscure way to many of the other sporadic simple groups in the monster. In [6] it is shown that if G is any finite subgroup of Conway's group $\mathrm{Aut}(\Lambda)$, then the lattice Λ^G of vectors fixed by G has many of the properties of Λ. In particular if it has no roots then the reflection group of the Lorentzian lattice $\Lambda^G \oplus U$ has a norm 0 Weyl vector ρ. This suggests the following question.

Problem. Let G be a finite subgroup of $\mathrm{Aut}(II_{25,1})$ fixing $\rho \in II_{25,1}$, and let L be the Lorentzian lattice of vectors fixed by G. Is there a "nice" Lie algebra with root lattice L? In particular, some experiments suggest the following conjecture.

Conjecture. Let g be an element of $M_{24} \subset \mathrm{Aut}(\Lambda)$ of order h with cycle shape 1^{24}, $1^8 2^8$, $1^6 3^6$, $1^4 5^4$, $1^2 2^2 3^2 6^2$, $1^3 7^3$, $1^2 11^2$, $1^1 2^1 7^1 14^1$, $1^1 3^1 5^1 15^1$, or $1^1 23^1$. Let Λ^g be the lattice of vectors of Λ fixed by g and let L be the Lorentzian lattice $\Lambda^g \oplus U$. If g has cycle shape $1^{n_1} 2^{n_2} \ldots$ let $\Delta_g(q)$ be the modular form $\eta(q)^{n_1}\eta(2q)^{n_2} \ldots = \sum_n \tau_g(n) q^n$.

Then we conjecture that

1 Λ^g has no roots. By the results of [6] this implies that the reflection group of L has a norm 0 Weyl vector ρ (i.e., ρ has inner product $-(r^2/2)$ with every simple root r of the reflection group).

2 The modular form $\Delta_g(q)$ has multiplicative coefficients and is the cusp form of smallest weight for the normalizer $\Gamma_0(h)+$ of the subgroup $\Gamma_0(h)$ of $PGL_2^+(Q)$.

3 Let M_g be the generalized Kac-Moody algebra whose real simple roots are those of the reflection group of L, and whose imaginary simple roots are the multiples $n\rho$ of ρ, with multiplicity equal to the number of fixed points of g^n, so that $\Delta_g(q) = q\prod_n(1-q^n)^{\mathrm{mult}(n\rho)}$. Define p_g by $\sum p_g(n)q^n = 1/\Delta_g(q)$. Then the root r of L has a multiplicity given by some expression involving p_g, possibly

$$\sum_{d|h,(r,L)} p_g(-r^2/d)$$

where the sum is over all positive integers d dividing both h and all the numbers $(r,s), s \in L$. This seems to be the simplest expression that gives the correct multiplicity for positive roots, multiples of ρ, and $II_{25,1}$. It does not work if g is one of the elements of M_{24} of cycle shape $1^4 2^2 4^4$ or $1^2 2^1 4^1 8^2$.

Conway and Norton [10] showed that there was a natural correspondence between automorphisms of Λ and certain conjugacy classes in the monster simple group, and these elements of the monster often have centralizers which are almost sporadic simple groups. For example, the elements of M_{24} of cycle shapes 1^{24}, $1^8 2^8$, $1^6 3^6$, $1^4 5^4$ and $1^3 7^3$ correspond in this way to the monster, the baby monster, the Fischer group Fi_{24}', the Harada Norton group, and the Held group. This suggests that the Lie algebras associated to these elements of M_{24} might be connected in some way to these simple groups. However, there is no known direct connection even between the monster Lie algebra and the monster simple group.

There also seems to be an interesting Lie superalgebra. Let U be the nonintegral lattice spanned by two vectors ρ, ρ' with $\rho^2 = \rho'^2 = 0, (\rho, \rho') = -1/2$, and let L be the direct sum of U and the E_8 lattice. We define a generalized Kac-Moody superalgebra whose root lattice is L by letting its real simple roots be the norm 1 vectors r of L with $(\rho, r) = -1/2$, and letting its imaginary simple roots be the positive multiples of ρ, each with multiplicity 8, where the even multiples of ρ are even roots, and the odd multiples of ρ are odd (or "super") roots. Then we conjecture that the multiplicity of a nonzero root $r \in L$ is the coefficient of $q^{(1-r^2)/2}$ in the modular form

$$q^{1/2}\prod_{i>0}(1-q^i)^{-8}(1+q^{i-1/2})^8.$$

of level 2. This modular form has the same relation to the theta functions of odd unimodular lattices as $\Delta(q)$ has to the theta functions of even unimodular lattices. If we could prove that there was some generalized Kac-Moody superalgebra with these root multiplicities, then it would follow from an argument similar to the proof of theorem 1 that its simple roots are just the ones above.

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