Coxeter groups, Lorentzian lattices, and K3 surfaces. *25 June, 27 July 1998*

Internat. Math. Res. Notices 1998, no. 19, 1011–1031.
Richard E. Borcherds, *
D.P.M.M.S., 16 Mill Lane, Cambridge, CB2 1SB, England.
e-mail: reb@dpmms.cam.ac.uk
www home page www.dpmms.cam.ac.uk/~reb

Contents.

1. Introduction.

The main result of this paper describes the normalizer $N_{W_\Pi}(W_J)$ of a finite parabolic subgroup W_J of a (possibly infinite) Coxeter group W_Π. More generally we describe $N_{W_\Pi.\Gamma_\Pi}(W_J)$ where Γ_Π is a group of diagram automorphisms of the Coxeter diagram Π of W_Π. By taking Π to be Conway's Coxeter diagram of the reflection group of $II_{1,25}$ we compute the automorphism groups of some Lorentzian lattices and K3 surfaces.

In the case when W_Π is a finite Coxeter group (and $\Gamma_\Pi = 1$) the normalizer of W_J has been described by Howlett [H]. His result states that $N_{W_\Pi}(W_J)$ is a split extension $W_J.W'_J$, where $W'_J = W_\Omega.\Gamma_\Omega$ is in turn a split extension with W_Ω a Coxeter group and Γ_Ω a more mysterious group acting on Ω. Howlett showed by case by case analysis that if Π is connected (and W_Π is finite) then Γ_Ω is an elementary abelian 2-group and is a subgroup of $Aut(J)$. When W_Π is infinite the normalizer $N_{W_\Pi}(W_J)$ has a similar structure, except that the group Γ_Ω can be more complicated. Although there is still a canonical map from Γ_Ω to $Aut(J)$, the kernel can be non trivial, though it has finite cohomological dimension. The kernel is trivial in the case of finite W_Π considered by Howlett because any finite group of finite cohomological dimension must be trivial. For example, the case when $J = A_1$, $\Gamma_\Pi = 1$ has been done by Brink [Br], who showed that Γ_Ω is a free group (and therefore has cohomological dimension at most 1). We will extend Brink's result to Coxeter diagrams of arbitrary finite reflection groups. More precisely we construct a category Q_4 using Π and J and prove that the classifying space of this category is a classifying space of the group Γ_Ω. The main point about this category Q_4 is that it is often finite and can often be written down explicitly, in which case we can easily read off a presentation of Γ_Π. For example, if $J = A_1$ we show that the classifying space of this category Q_4 is 1-dimensional, so its fundamental group is free and we recover Brink's result. After writing this paper I discovered that Brink and Howlett had previously announced a related description of the normalizer of a parabolic subgroup of a Coxeter group; see [B-H], and see example 2.8 for the relation between their result and theorem 2.7.

* Supported by a Royal Society professorship and an NSF grant.

For later applications we need some generalizations as follows. First of all, instead of calculating the normalizer of W_J in a Coxeter group W_Π, we calculate the normalizer in an extension $W_\Pi.\Gamma_\Pi$, where Γ_Π is a group of diagram automorphisms. Secondly, we sometimes want to compute not the full normalizer, but a subgroup with image contained in some subgroup Γ_J of $Aut(J)$. Thirdly, we sometimes want to vary the choice of the Coxeter group W_Ω, which we do by varying a certain normal subgroup R of Γ_J. For example, in calculating the automorphism groups of $K3$ surfaces we take W_Ω to be generated by reflections of norm -2 vectors rather than by all reflections, so we take $R = 1$.

Section 2 contains a statement of the main result (theorem 2.7) describing a classifying category for the group Γ_Ω, and section 3 contains the proof of this result. Section 4 contains some more information about the structure of Γ_Ω. In section 5 we give some applications of theorem 2.7, and in particular show how to describe the automorphism groups of some Lorentzian lattices by embedding them in $II_{1,25}$ and using the description of $Aut(II_{1,25})$ in [C]. The idea of studying Lorentzian lattices by embedding them as orthogonal complements of root lattices in $II_{1,25}$ comes from Conway and Sloane ([C-S]).

Work of I. Piatetski-Shapiro and I. R. Shafarevich [P-S] shows that there is a map from the automorphism group of a K3 surface to the group of automorphisms of its Picard lattice modulo the group generated by reflections of norm -2 vectors which has finite kernel and co-finite image, so in practice if we want to describe the automorphisms of a K3 surface the main step is to calculate the automorphism group of its Picard lattice. Kondo showed in [K] that the automorphism groups of some K3 surfaces could be studied by embedding their Picard lattice as the orthogonal complement of a root lattice in $II_{1,25}$. We use Kondo's idea to describe the automorphism groups of some K3 surfaces in terms of combinatorics of the Leech lattice. In particular we reprove some results of Vinberg [V] on the "most algebraic" K3 surfaces and extend them to the "next most algebraic" K3 surface. Kondo showed in [K] that the automorphism group of the Kummer surface of a generic genus 2 Jacobian was generated by the classically known automorphisms together with some new automorphisms found by Keum [Ke], and we show how to use Kondo's results to describe the structure of this group. Kondo and Keum [K-K] have recently proved similar results for some Kummer surfaces associated to the products of two elliptic curves.

Kondo recently found another mysterious connection between automorphism groups of K3 surfaces and Niemeier lattices [K98], and used this to give a short proof of Mukai's classification [Mu] of the finite groups that act on K3 surfaces.

I would like to thank I. Cherednik, I. Grojnowski, R. B. Howlett, J. M. E. Hyland, S. Kondo, U. Ray, and G. Segal for their help.

2. Notation and statement of main theorem.

This section states the main result (theorem 2.7) describing normalizers of parabolic subgroups of Coxeter groups.

We recall some basic definitions about Coxeter systems. For more about them see [Hi] or [Bo]. A pair (W, S) is called a **Coxeter system** if W is a group with a subset S such that W has the presentation

$$\langle s : s \in S | (ss')^{m_{ss'}} = 1 \text{ when } m_{ss'} < \infty \rangle$$

where $m_{ss'} \in \{1, 2, 3, \ldots, \infty\}$ is the order of ss', and $m_{ss'} = 1$ if and only if $s = s'$. A diagram automorphism of S is an automorphism of the set S that extends to an automorphism of the group W, and $Aut(S)$ means the group of diagram automorphisms of S. We say that (W, S) is **irreducible** if S is not a union of two disjoint commuting subsets. The number of elements of S is called its **rank**. The Coxeter system is called **spherical** if W has finite order. The irreducible spherical Coxeter diagrams are A_n $(n \geq 1)$, $B_n = C_n$ $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, $G_2^{(n)} = I_2(n)$ $(n \geq 5)$, H_3, and H_4. It is also sometimes useful to define the Coxeter diagrams $B_1 = C_1 = A_1$, $D_3 = A_3$, $D_2 = A_1^2$, $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2A_1$, $G_2^{(4)} = B_2 = C_2$, $G_2^{(3)} = A_2$, $G_2^{(2)} = A_1^2$.

If (W_Π, Π) is a Coxeter system then we write V_Π for the (possibly infinite dimensional) real vector space with a basis of elements e_s for $s \in \Pi$, and put a symmetric bilinear form on V_Π by defining

$$(e_s, e_{s'}) = 2\cos(\pi/m_{ss'}).$$

Note that we normalize the roots e_s so that they have norm $(e_s, e_s) = -2$ rather than 1; this is done to be consistent with the usual conventions in algebraic geometry.

The Coxeter group W_Π acts on V_Π with the element $s \in \Pi \subseteq W_\Pi$ acting as the reflection $v \mapsto v + (v, e_s)e_s$ in the hyperplane $e_s^\perp$. Any subgroup Γ_Π of $Aut(\Pi)$ acts on V_Π by permutations of the elements e_s, so we get an action of $W.\Gamma_\Pi$ on V_Π, and hence on the dual space V_Π^*. We write Δ^+ for the set of positive roots of W_Π. We define the fundamental chamber $C_\Pi \subseteq V_\Pi^*$ of W_Π by

$$C_\Pi = \{x \in V_\Pi^* | x(r) \geq 0 \text{ for all } r \in \Pi \text{ and } x(r) > 0 \text{ for almost all } r \in \Delta^+\}.$$

(Recall that "almost all" means "all but a finite number of".) A theorem due independently to Tits and Vinberg states that no two distinct points of C_Π are conjugate under W_Π, and the subgroup of W_Π fixing all points of some subset A of W_Π is generated by the reflections in the faces of C_Π containing A. In particular W_Π acts simply transitively on the conjugates of C_Π. The union $W_\Pi(C_\Pi)$ of all conjugates of C_Π under W_Π is given by

$$W_\Pi(C_\Pi) = \{x \in V_\Pi^* | x(r) > 0 \text{ for almost all } r \in \Delta^+\}.$$

In particular $W_\Pi(C_\Pi)$ is convex and closed under multiplication by positive real numbers. If $x \in W_\Pi(C_\Pi)$ then the set of roots vanishing on x is a finite root system.

Note that $W_\Pi(C_\Pi)$ is usually slightly smaller than the Tits cone, which is defined in the same way except that we omit the condition that $x(e_s) > 0$ for all but a finite number of s in the definition of the fundamental domain. The Tits cone can be thought of as obtained from $W_\Pi(C_\Pi)$ by "adding some boundary components". The reason for using $W_\Pi(C_\Pi)$ rather than the Tits cone is that the cone $W_\Pi(C_\Pi)$ has the property that the subgroup of W_Π fixing any vector of it is finite.

We fix a spherical subset J of Π. In particular we get a spherical Coxeter system (W_J, J). Suppose K is an isometry of Coxeter diagrams from J into Π. We write W_K for the finite reflection group generated by $K(J)$. There is a natural homomorphism $p : N_{W_\Pi.\Gamma_\Pi}(W_K) \mapsto Aut(J)$. We let $N_{W_\Pi.\Gamma_\Pi}(W_K; \Gamma_J)$ be the subgroup of elements whose image is in a subgroup Γ_J of $Aut(J)$. We are interested in describing the group $N_{W_\Pi.\Gamma_\Pi}(W_J;\Gamma_J)$. Most of the time we take $\Gamma_J = Aut(J)$ in which case $N_{W_\Pi.\Gamma_\Pi}(W_J;\Gamma_J) = N_{W_\Pi.\Gamma_\Pi}(W_J)$, but it is occasionally useful to use other values of Γ_J; see example 5.7 and theorem 4.1.

We define W'_K to be the subgroup of $N_{W_\Pi.\Gamma_\Pi}(W_K;\Gamma_J)$ mapping $K(J)$ to itself.

Lemma 2.1. $N_{W_\Pi.\Gamma_\Pi}(W_K;\Gamma_J) = W_K.W'_K$.

Proof. This follows immediately from the fact that W_K acts simply transitively on the Weyl chambers of W_K, and W'_K is the subgroup of $N_{W_\Pi.\Gamma_\Pi}(W_K;\Gamma_J)$ fixing a Weyl chamber of W_K. This proves lemma 2.1.

The group W'_K acts on the subspace $V_\Pi^{*W_K}$ of V_Π^* of all vectors fixed by W_K. We now construct a reflection group W_{Ω_K} acting on $V_\Pi^{*W_K}$. We choose a normal subgroup R of Γ_J. (The subgroup R is used to control the reflection group W_Ω defined below. Often we want W_Ω to be as large as possible and we take $R = \Gamma_J$, but sometimes we want to take a smaller W_Ω; see examples 5.3, 5.4 and 5.5.) We define the group W_{Ω_K} to be the subgroup of $W_\Pi \cap W'_K$ generated by elements $w \in W_\Pi \cap W'_K$ such that w acts on $V_\Pi^{*W_K}$ as a reflection and acts on J as an element of R. We define Ω_K to be the Coxeter diagram of W_{Ω_K} and C_{Ω_K} to be its fundamental chamber. If K is the identity map from J to $J \subseteq \Pi$ then we write W_Ω, C_Ω, and Ω instead of W_{Ω_K}, C_{Ω_K}, and Ω_K. Note that W_{Ω_K} is obviously contained in the inverse image of R in $W_\Pi \cap W'_K$, but can be much smaller; see for example the discussion of D_4 in example 5.7.

We define the group Γ_{Ω_K} to be the subgroup of W'_K of elements w with $w(C_{\Omega_K}) = C_{\Omega_K}$.

Lemma 2.2. *The group W'_K is a semidirect product $W'_K = W_{\Omega_K}.\Gamma_{\Omega_K}$.*

Proof. We first show that the group W_{Ω_K} acts faithfully on $V_\Pi^{*W_K}$. More generally we will show that if $w \in W_\Pi \cap W'_K$ acts trivially on $V_\Pi^{*W_K}$, then $w = 1$. To see this we observe that w fixes the point $x \in C_\Pi \cap V_\Pi^{*W_K}$ such that $x(e_s) = 0$ if $s \in K(J)$ and $x(e_s) = 1$ if $s \notin K(J)$. Therefore w is in the subgroup W_K of W_Π generated by the simple reflections of W_Π fixing x. On the other hand w maps $K(J)$ into itself as $w \in W'_K$. This implies that $w = 1$ because 1 is the only element of W_K mapping $K(J)$ into itself. This proves that the group W_{Ω_K} acts faithfully on $V_\Pi^{*W_K}$.

Lemma 2.2 now follows from the fact that W_{Ω_K} acts simply transitively on the conjugates of C_{Ω_K} under W'_K, and Γ_{Ω_K} is the stabilizer of C_{Ω_K}. This proves lemma 2.2.

Warning: the group Γ_{Ω_K} need not act faithfully on $V_\Pi^{*W_K}$ (though it does act faithfully on $V_\Pi^{*W_K} \times J$).

We define a **classifying category** of a group Γ to be a category whose geometric realization is a classifying space for Γ. (Recall from [Q] that the geometric realization of a category is a space with a 0-cell for each object and an n-cell for each sequence of n composable morphisms if $n > 0$.) For example, the category with one object whose morphisms are the elements of Γ (with composition given by group multiplication) is a classifying category for Γ.

We have more or less reduced the problem of describing $N_{W_\Pi.\Gamma_\Pi}(W_J;\Gamma_J)$ to that of describing Γ_{Ω_K}. (The Coxeter diagram Ω of W_Ω can be described once we know Γ_Ω.) The main theorem of this paper describes Γ_{Ω_K} by giving an explicit classifying category for it. To define this category we need some more definitions.

Suppose that S is the Coxeter diagram of a finite reflection group G of a finite dimensional vector space with no vectors fixed by G. Fix a Weyl chamber C of G, so that the walls of C correspond to the points of S. Then there is a unique element σ_S of G taking C to $-C$ called the opposition involution. The involution $-\sigma_S$ acts on the the Coxeter diagram S, and its action on S does not depend on the choice of C. This action can be described as follows. The points of the Coxeter diagram correspond to the simple roots of C. This set of roots is the same as the set of simple roots of $\sigma_S(C) = -C$ multiplied by -1. Hence $-\sigma_S$ acts on this set of simple roots, in other words on the Coxeter diagram S. The involution $-\sigma_S$ of S can be described explicitly as follows. On diagrams of type A_1, $B_n = C_n$, D_{2n}, E_7, E_8, F_4, $G_2^{(2n)}$, H_3, H_4 for any $n \geq 1$ the involution $-\sigma_S$ is the trivial automorphism of S, while for diagrams of types A_{n+1}, D_{2n+1}, E_6, $G_2^{(2n+1)}$ for $n \geq 1$ the involution $-\sigma_S$ is the unique nontrivial automorphism of the Coxeter diagram S. Finally if the diagram S is a union of connected components then $-\sigma_S$ acts on each connected component as described above.

Suppose that J and S are Coxeter diagrams. Suppose that K and K' are two isometries from J into S. We define K and K' to be **adjacent** if there is a point s of S not in $K(J)$ such that $K(J) \cup s$ is spherical and $\sigma_{K(J)\cup s}\sigma_{K(J)}$ takes K to K'. If K is adjacent to K' then K' is adjacent to K. We define two isometries K, K' of J into S to be **associate** if there is a sequence of isometries $K = K_1, K_2, \dots, K_n = K'$ from J into S such that K_i and K_{i+1} are adjacent for $1 \leq i < n$. We write $\overline{K}$ for the equivalence class of all isometries from J into S that are associate to K.

Remark. The term associate is closely related to the same term in algebraic group theory, where two parabolic subgroups are called associate if their Levi factors are conjugate. There is a small difference because a standard parabolic subgroup corresponds to a subdiagram of the Dynkin diagram, while we work with *labeled* subdiagrams of a Coxeter diagram.

Suppose that S is a spherical Coxeter diagram. Recall that R is a normal subgroup of $\Gamma_J \subseteq Aut(J)$. We define an isometry $K : J \mapsto S$ to be **R-reflective** if K is adjacent to $r(K)$ for some $r \in R$. We define the equivalence class $\overline{K}$ to be **R-reflective** if at least one of its elements is R-reflective. (The reason for the name "R-reflective" appears in lemmas 3.8 and 3.9.)

Example 2.3. Suppose J is A_1 and S is A_3, with the isometries from J into S labeled as K_1, K_2, K_3 in the obvious way. Then the isometry K_1 is adjacent to K_2 as $\sigma_{K_1(J)\cup K_2(J)}\sigma_{K_1}$ takes K_1 to K_2. Similarly K_2 is adjacent to K_3, and K_1 is associate to K_3 but not adjacent to K_3. The isometries K_1 and K_3 are adjacent to themselves and K_2 is not adjacent to itself, so the isometry K_2 is not R-reflective but the isometries K_1 and K_3 are. So K_2 is not R-reflective but the equivalence class $\overline{K_2}$ is.

Example 2.4. Suppose K is D_5 and S is D_6. Then there are exactly two isometries $K_1, K_2 : J \mapsto S$ of S, which are adjacent to each other but not to themselves. These two isometries are exchanged by the nontrivial automorphism of D_5. Hence if R contains just $1 \in Aut(D_5)$ then K_1, K_2, and the equivalence class $\overline{K_1} = \{K_1, K_2\}$ are not R-reflective, but if R is the whole of $Aut(D_5) = Z/2Z$ then all of them are R-reflective.

Example 2.5 Suppose K is A_3 and S is D_5. Then there are 8 isometries $K : J \mapsto S$. These form two equivalence classes under the relation of being associate, one of size 2 and one of size 6. This shows that two isometries from a connected diagram J into S need not be associate to conjugates of each other under $Aut(J)$.

Example 2.6 Suppose J is A_2. If S is A_n for $n \geq 2$ then there are two equivalence classes of isometries $K : J \mapsto S$, which are exchanged by the nontrivial automorphism of A_2. However if S is D_n ($n \geq 4$), E_6, E_7, or E_8 then there is only one equivalence class, as the A_2 can be reversed by doing a "three point turn" around the point of valence 3 in S.

We define a poset P_3^+ as follows. The objects of P_3^+ are pairs $(S, \overline{K})$ consisting of a spherical subdiagram S of Π and an equivalence class $\overline{K}$ of isometries from J into S. We define the partial order on P_3^+ by putting $(S, \overline{K}) \leq (S', \overline{K'})$ if $S \subseteq S'$ and $\overline{K} \subseteq \overline{K'}$. We define P_3 to be the sub-poset of P_3^+ of elements $(S, \overline{K})$ such that $\overline{K}$ is not R-reflective.

Note that the condition that $\overline{K}$ is not R-reflective is quite restrictive and implies that S is usually not much larger than J and in any case has at most twice the rank of J. In particular S cannot contain any root orthogonal to $K(J)$ as this implies that K is R-reflective for any R.

Suppose that P is a poset acted on by a group G. We define the **homotopy quotient** of P by G to be the following category Q. The objects of Q are the elements of P. The morphisms from $p_1 \in P$ to $p_2 \in P$ correspond to the group elements $g \in G$ such that $g(p_1) \leq p_2$, and composition of morphisms is given by multiplication of group elements. If G is trivial this is the usual category associated to the poset P, and if P has just one point this is the usual category with one object associated to the group G. If we take a full set of representatives of the orbits of G on P, then the full sub category of Q with these objects is a skeleton of the category Q.

The poset P_3 is acted on by the group $\Gamma_J \times \Gamma_\Pi \subseteq Aut(J) \times Aut(\Pi)$. We define Q_3 to be the homotopy quotient of P_3 by $\Gamma_J \times \Gamma_\Pi$, and we construct the category Q_4 as a skeleton of Q_3 as above. In other words the objects of Q_4 are a complete set of representatives for the orbits of $\Gamma_J \times \Gamma_\Pi$ on the elements of P_3 and the morphisms from $(S, \overline{K})$ to $(S', \overline{K'})$ correspond to group elements $\gamma \in \Gamma_J \times \Gamma_\Pi$ such that $\gamma((S, \overline{K})) \leq (S', \overline{K'})$.

The main result of this paper is the following description of the classifying space of Γ_Ω.

Theorem 2.7. *Suppose we are given the following objects.*

(W_Π, Π) *A Coxeter system.*
Γ_Π *A subgroup of* $Aut(\Pi)$.
(W_J, J) *A spherical Coxeter system with* $J \subseteq \Pi$.
Γ_J *A subgroup of* $Aut(J)$.
R *A normal subgroup of* Γ_J.

Define W_Ω, Γ_Ω, *and* Q_4 *as above, so that the group* $N_{W_\Pi.\Gamma_\Pi}(W_J; \Gamma_J)$ *has the structure*

$$W_J.W_\Omega.\Gamma_\Omega$$

where W_Ω *is a Coxeter group. Then the component of* Q_4 *containing the object* $(J, \overline{id_J})$ *is a classifying category for the group* Γ_Ω.

Theorem 2.7 gives a presentation of the group Γ_Ω because Γ_Ω is the fundamental group of the category Q_4 with respect to the basepoint $(J, \overline{id_J})$, and it is easy to write down a presentation of the fundamental group of any connected category Q as follows.

Choose a spanning tree T for the underlying 1-complex of Q (which has a point for each object of Q and a 1-cell for each morphism). Then the fundamental group Γ_Ω of Q has a presentation as follows. The group Γ_Ω has a generator $\overline{g}$ for each morphism g of Q. The relations are $\overline{gh} = \overline{g}\overline{h}$ whenever gh is defined, and $g = 1$ for g in the spanning tree.

Example 2.8. Suppose we put $R = \Gamma_J = Aut(J)$ and $\Gamma_\Pi = 1$. Then we see from theorem 2.7 that $N_{W_\Pi}(W_J) = W_J.W_\Omega.\Gamma_\Omega$, where Γ_Ω is the fundamental group of the component of Q_4 corresponding to J. So in particular theorem 2.7 describes normalizers of finite parabolic subgroups of Coxeter groups. More generally, Brink and Howlett [B-H] have described a presentation for normalizers of possibly infinite parabolic subgroups of Coxeter groups. It is not trivial to see that the presentation given by [B-H] is equivalent to the one given by theorem 2.7 (though of course this follows from the fact that they are both presentations of the same group). Howlett pointed out to me that their result only requires considering subdiagrams S of Π whose rank is at most $2 + rank(J)$ to get the relations of Γ_Ω, and of rank at most $1 + rank(J)$ to get the generators of Γ_Ω. It seems possible that a similar simplification could be made to theorem 2.7 if all that is required is a presentation rather than a classifying space. Perhaps the natural map from $\pi_i(Q_4^j)$ to $\pi_i(Q_4)$ is an isomorphism for $i < j$ and an epimorphism for $i = j$, where Q_4^j is the full sub-category of Q_4 whose objects are the elements $(S, \overline{K})$ such that $rank(S) \leq rank(J) + j$. If so, the map from $\pi_1(Q_4^2)$ to $\pi_1(Q_4)$ would be an isomorphism, so this would give a closer connection to the presentation of Brink and Howlett. Their result also suggests that theorem 2.7 could be generalized by allowing W_J to be infinite and modifying the definition of Q_4 to allow subdiagrams S such that W_J has finite index in W_S.

3. Proof of main theorem.

This section gives the proof of the theorem 2.7. The idea of the proof is to construct categories and functors according to the following diagram.

$$Q_1 \longleftarrow Q_2 \longrightarrow Q_3 \longleftarrow Q_4$$

It is easy to show that a component of Q_1 is a classifying category for Γ_Ω. We also show that the functors between the categories are all homotopy equivalences, so a component of Q_4 is a classifying category for Γ_Ω, which is what we wanted to prove.

We define an isometry from J into the roots of W_Π to be **primitive** if it is conjugate under W_Π to an isometry from J into Π. An example of a non-primitive isometry is an isometry from A_1^4 into the roots of D_4.

We define a category Q_1 as follows. We define the poset P_1 to be the poset of pairs (C, K) where K is a primitive isometry from J into the (possibly non-simple) roots of W_Π, and C is a Weyl chamber of the reflection group W_{Ω_K} of $V_\Pi^{*W_K}$. The partial order on P_1 is the trivial one with $(C_1, K_1) \leq (C_2, K_2)$ if and only if $(C_1, K_1) = (C_2, K_2)$. The objects of P_1 are acted on in the natural way by the group $W_\Pi.\Gamma_\Pi$ via its action on V_Π, and by the group Γ_J via its action on J. We define the category Q_1 to be the homotopy quotient of P_1 by the group $\Gamma_J \times W_\Pi.\Gamma_\Pi$.

Lemma 3.1. *The component of Q_1 containing the object $id_J : J \mapsto J \subseteq \Pi$ is a classifying category for the group Γ_Ω.*

Proof. This follows immediately from the fact that Q_1 is a groupoid such that the automorphism group of the object K is the group Γ_{Ω_K}. This proves lemma 3.1.

Let C_Π be the Weyl chamber of W_Π defined in section 2. By a face of C_Π we mean a nonempty intersection of C_Π with some of the hyperplanes bounding C_Π. The faces of C_Π of codimension n correspond to the spherical subdiagrams of Π of rank n. We define a Π-cell to be a conjugate of a face of C_Π under W_Π. The cone X is the union of all Π-cells, and the intersection of two Π-cells is either empty or another Π-cell.

We define a category Q_2 and posets P_2, P_2^+ as follows. The objects of the poset P_2^+ are the pairs (D, K) where K is a primitive isometry from J into the roots of W_Π, and D is a Π-cell contained in $V_\Pi^{*W_K}$. We define the partial order on P_2^+ by saying $(D_1, K_1) \leq (D_2, K_2)$ if $D_1 \subseteq D_2$ and $K_1 = K_2$. We define P_2 to be the sub-poset of P_2^+ of elements (D, K) such that D is not contained in a reflection hyperplane of W_{Ω_K}. The category Q_2 is defined to be the homotopy quotient of P_2 by $\Gamma_J \times W_\Pi.\Gamma_\Pi$.

Lemma 3.2. *Suppose G is a group, P_1 and P_2 are G-posets, and f is a morphism of G-posets from P_2 to P_1. Also suppose that for any $Y \in P_1$ the poset $f^{-1}(Y)$ is contractible (in other words the corresponding simplicial complex is contractible). Then the functor induced by f between the homotopy quotient categories Q_2, Q_1 of the posets P_2 and P_1 by the group G is a homotopy equivalence.*

Proof. If f is a functor from a category Q_2 to a category Q_1 and Y is an object of Q_1 then we write $f^{-1}(Y)$ for the fiber of f over Y, in other words the sub category of Q_2 whose morphisms are those mapped to the identity of Y by f. We write $Y\backslash f$ for the category consisting of pairs (X, v) with $v : Y \mapsto f(X)$, where a morphism from (X, v) to (X', v') is a morphism $w : X \mapsto X'$ such that $f(w)v = v'$. Then a result due to Quillen (the corollary to theorem A on page 9 of [Q]) states that f is a homotopy equivalence provided that for all Y in Q_1 the poset $f^{-1}(Y)$ is contractible and the functor from $f^{-1}(Y)$ to $Y\backslash f$ taking X to (X, id_Y) has a right adjoint. (Here id_Y is the identity morphism of Y.)

We will use Quillen's result to show that f is a homotopy equivalence. For any object Y of P_1 the category $f^{-1}(Y)$ is just the category of the poset $f^{-1}(Y)$, which is contractible by assumption. So it only remains to check the condition about the existence of a right adjoint from $Y\backslash f$ to $f^{-1}(Y)$. The category $Y\backslash f$ has as objects pairs (X, v) with $v \in G$, $v(Y) \leq f(X)$ and there is a morphism from (X, v) to (X', v') if and only if $v^{-1}(X) = v'^{-1}(X')$, in which case the morphism is unique. We define a functor g from $Y\backslash f$ to $f^{-1}(Y)$ on objects by $g((X, v)) = v^{-1}(X)$. It is easy to check that this extends in a unique way to morphisms. It is a right adjoint to f because $Y \leq g((X, v))$ if and only if there is a morphism (necessarily unique) from $f(Y)$ to (X, v), both conditions being equivalent to $v(Y) \leq X$. This shows that the conditions of Quillen's result are satisfied, so f is a homotopy equivalence. This proves lemma 3.2.

Lemma 3.3. *The functor f is a homotopy equivalence from Q_2 to Q_1.*

Proof. By lemma 3.2 it is sufficient to check that for each $Y \in P_1$, the sub poset $f^{-1}(Y)$ of P_2 is contractible. The poset $f^{-1}(Y)$ is the poset of a cell decomposition of a convex cone in a real vector space. As any convex set is contractible, the poset $f^{-1}(Y)$ is also contractible. This proves lemma 3.3.

Lemma 3.4. *Suppose (W, S) is a spherical Coxeter system acting on the vector space V_S with Weyl chamber C. Suppose K is an isometry from J into S. Let $V_S^{*W_K}$ be the subspace of V^* fixed by W_K, where W_K is the reflection group whose simple roots are the points $K(J)$. The walls of $C \cap V_S^{*W_K}$ correspond to the points in S not in the image of K; let s be one of these points and let $s^\perp \cap V_S^{*W_K}$ be the wall in $V_S^{*W_K}$ corresponding to s. Choose $w \in W$ so that $w(C)$ is the (unique) Weyl chamber of W such that $w(C) \cap V_S^{*W_K}$ is the cell in $V_S^{*W_K}$ on the other side of $s^\perp \cap V_S^{*W_K}$ to $C \cap V_S^{*W_K}$ and such that C and $w(C)$ are both in the same Weyl chamber of W_K. Then*

$$w = \sigma_{K(J)\cup s}\sigma_{K(J)}.$$

Proof. We can reduce to the case when $S = K(J) \cup s$, so that $V_S^{*W_K}$ is one dimensional and $s^\perp \cap V_S^{*W_K}$ is just the point 0. Then $\sigma_S(\sigma_{K(J)}(C)) = -\sigma_{K(J)}(C)$ which contains $-C \cap V_S^{*W_K}$, so $\sigma_S(\sigma_{K(J)}(C)) \cap V_S^{*W_K}$ is a cell on the other side of $s^\perp \cap V_S^{*W_K}$ to $C \cap V_S^{*W_K}$. Moreover $\sigma_{K(J)}(C)$ is in the opposite Weyl chamber of $K(J)$ to C, and $\sigma_S(\sigma_{K(J)}(C))$ is in the opposite Weyl chamber to $\sigma_{K(J)}(C)$, so $\sigma_S(\sigma_{K(J)}(C))$ is in the same Weyl chamber of $K(J)$ as C. This shows that the element w of the lemma is $\sigma_S\sigma_{K(J)}$. This proves lemma 3.4.

A result similar to the following lemma (using subsets of S rather than isometries $K : J \mapsto S$) is given in [H, lemma 5] when S is finite and in [D] for arbitrary S.

Lemma 3.5. *Suppose (W, S) is a Coxeter system, J is a spherical Coxeter diagram, and K and K' are two isometries from J into S. Then K and K' are conjugate under W if and only if they are associate.*

Proof. First suppose that K and K' are adjacent. Then $\sigma_{K\cup K'}(\sigma_K(K)) = K'$, so K and K' are conjugate under W. Next suppose K and K' are associate. Then by definition we can find a sequence $K = K_1, K_2, \ldots, K_n = K'$ such that K_i and K_{i+1} are adjacent for all i. Hence $K = K_1$ and $K' = K_n$ are also conjugate under W. So associate isometries from J into S are conjugate under W.

Conversely, suppose that K and K' are conjugate by an element $w \in W$. Consider the subspace $V_S^{*W_K}$ of V_S^*, and the codimension 0 cells in it of the form $V_S^{*W_K} \cap C$ for some Weyl chamber C of W. For any two such cells, for example $D = V_S^{*W_K} \cap C$ and $D' = V_S^{*W_K} \cap w(C)$, we can find a sequence $D = D_1, D_2, \ldots, D_n = D'$ such that D_i and D_{i+1} are adjacent by a face of codimension 1 in $V_S^{*W_K}$. For each i let C_i be the (unique) Weyl chamber whose intersection with $V_S^{*W_K}$ is D_i and that is contained in the Weyl chamber of W_K. We identify each C_i with C_Π using w_i. The set $K(J)$ is a subset of the simple roots of C_i, so $K_i = w_i^{-1}(K)$ maps J to the simple roots of C_Π. The isometries K_i and K_{i+1} are adjacent for all i, because the element $w_{i+1}w_i^{-1}$ mapping C_i to C_{i+1} is equal to $\sigma_{K(J)\cup s_i}\sigma_{K(J)}$, where s_i is the simple root of C_i orthogonal to $D_i \cap D_{i+1}$ but not to D_i. Therefore $K = K_1$ and $K' = K_n$ are adjacent. This proves lemma 3.5.

Suppose that P is a W-poset for a group W with the property that if $p \le w(p)$ for $w \in W, p \in P$ then $p = w(p)$. We define the **quotient** $W\backslash P$ of P by W to be the poset whose elements are the orbits Wp of W acting on P, where we put $Wp \le Wq$ if $w(p) \le q$ for some $w \in W$. This should not be confused with the homotopy quotient of P by W.

Lemma 3.6. *The $\Gamma_J \times \Gamma_\Pi$ posets P_3^+ and $W_\Pi \backslash P_2^+$ are isomorphic.*

Proof. We will construct an isomorphism f of posets from $W_\Pi \backslash P_2^+$ to P_3^+. Suppose (D, K) is an element of P_2^+ representing an element of $W_\Pi \backslash P_2^+$. We can find an element w of W_Π such that $w(D) \subseteq C_\Pi$ and $w(K(J)) \subseteq \Pi$. We define $f((D, K))$ to be $(S, \overline{w(K)})$, where S is the set of simple roots of C_Π orthogonal to $w(D)$. We check that this is well defined even though w is not unique. To prove this we can assume that $D \subseteq C_\Pi$ and $K(J) \subseteq \Pi$. Then the different possibilities for w are elements of the group generated by the reflections fixing D and the Weyl chamber of $K(J)$. But these elements take K to an associated isometry $K : J \mapsto S$, so the equivalence class $\overline{K}$ is well defined by lemma 3.5. These elements also take S to S, so S is well defined. This proves that $(S, \overline{K})$ is uniquely defined.

The isomorphism f of posets from $W_\Pi \backslash P_2^+$ to P_3^+ obviously preserves the $\Gamma_J \times \Gamma_\Pi$ action on both posets. This proves lemma 3.6.

Lemma 3.7. *Suppose (W, S) is a spherical Coxeter system, K is an isometry $K : J \mapsto S$, and s is a point of S not in $K(J)$. Then there is an element of W mapping $V_S^{*W_K}$ to itself and acting on $V_S^{*W_K}$ as reflection in $s^\perp \cap V_S^{*W_K}$ if and only if $-\sigma_{K(J)\cup s}$ maps $K(J)$ to itself. If such an element of W exists, then there is a unique such element w mapping K to itself, given by $w = \sigma_{K(J)\cup s}\sigma_{K(J)}$.*

Proof. If an element of w maps $V_S^{*W_K}$ to itself then there is a unique element of W with the same action on $V_S^{*W_K}$ and mapping $K(J)$ to itself because W_K acts simply transitively on its Weyl chambers, so we may assume that w maps $K(J)$ to itself. If in addition w acts on $V_S^{*W_K}$ as reflection in $s^\perp \cap V_S^{*W_K}$ then by lemma 3.4 w must be $\sigma_{K(J)\cup s}\sigma_{K(J)}$.

Conversely if $-\sigma_{K(J)\cup s}$ maps $K(J)$ to itself then $\sigma_{K(J)\cup s}\sigma_{K(J)}$ maps $K(J)$ to itself and acts on $V_S^{*W_K}$ as reflection in $s^\perp \cap V_S^{*W_K}$. This proves lemma 3.7.

Lemma 3.8. *Suppose K is an isometry from J into a spherical subdiagram S of Π. Then $K : J \mapsto S$ is R-reflective if and only if the Π-cell $S^\perp \cap C_\Pi$ is contained in a reflection hyperplane of W_{Ω_K} of the form $s^\perp \cap V_\Pi^{*W_K}$ for $s \in \Pi$.*

Proof. First suppose that $K : J \mapsto S$ is R-reflective. Then there is a point $s \in S$ not in $K(J)$ such that $w = \sigma_{K(J)\cup s}\sigma_{K(J)}$ acts on $K(J)$ as an element of R. So w is a reflection of W_Ω corresponding to the hyperplane $s^\perp \cap V_\Pi^{*W_K}$, and this hyperplane contains $S^\perp \cap C_\Pi$.

Conversely, suppose that $S^\perp \cap C_\Pi$ is contained in a reflection hyperplane of $w \in W_{\Omega_K}$ of the form $s^\perp \cap V_\Pi^{*W_K}$ for $s \in \Pi$. Then we must have $s \in S$ because $S^\perp \cap C_\Pi \subseteq s^\perp$. The element w must be equal to $\sigma_{K(J)\cup s}\sigma_{K(J)}$, and this element acts on K as an element of R because $w \in W_\Omega$. Therefore $K : J \mapsto S$ is R-reflective. This proves lemma 3.8.

Lemma 3.9. *Suppose K is an isometry from J into a spherical subdiagram S of Π. Then the equivalence class $\overline{K}$ is R-reflective if and only if the Π-cell $S^\perp \cap C_\Pi$ of $V_\Pi^{*W_K}$ corresponding to S is contained in a reflection hyperplane of W_{Ω_K}.*

Proof. Suppose the cell of $V_\Pi^{*W_K}$ corresponding to S is contained in a reflection hyperplane of W_{Ω_K}. Choose a Weyl chamber C_Π for W_Π such that this reflection hyperplane is a wall of $C_\Pi \cap V_\Pi^{*W_K}$. Then by lemma 3.8 the corresponding isometry $K' : J \mapsto S$ is R-reflective, and by lemma 3.5 is associate to K. So $\overline{K}$ is R-reflective.

Conversely suppose that $\overline{K}$ is R-reflective. Then $w(K)$ is R-reflective and has image in S for some $w \in W_\Omega$. By lemma 3.8 this implies that $w(S^\perp \cap C_\Pi)$ is contained in a reflection hyperplane of W_Ω, so the same is true of $S^\perp \cap C_\Pi$. This proves lemma 3.9.

Lemma 3.10. *Suppose P is a G-poset for some group G. Suppose that W is a normal subgroup of G such that if $p \le q$ and $w(p) \le q$ for some $w \in W$, $p, q \in P$, then $w = 1$. Then the homotopy quotient Q_2 of P by G is equivalent to the homotopy quotient Q_3 of $W\backslash P$ by G/W.*

Proof. Recall from [M, theorem 1, page 91] that if f is any functor from a category Q_2 to a category Q_3 then f is an equivalence if the following two conditions are satisfied:

1 Any object of Q_3 is isomorphic to some object in the image of f.

2 For any two objects p, q of Q_2, f induces an isomorphism from $Mor(p,q)$ to $Mor(f(p), f(q))$.

We will apply this to show that our categories Q_2 and Q_3 are equivalent. We define f on objects by $f(p) = Wp$, and define f on morphisms using the obvious homomorphism from G to G/W. Condition 1 above is satisfied because every element of P_3 is the image of an element of P_2, so every object of Q_3 is the image of an object of Q_2. Suppose p and q are objects of P_2. The set of morphisms of Q_2 from p to q can be identified with the set of elements g of G such that $g(p) \le q$, and $Mor_{Q_3}(f(p), f(q))$ can also be identified with the set of elements g of G such that $g(p) \le q$, so condition 2 above is satisfied. This shows that f is an equivalence and proves lemma 3.10.

Lemma 3.11. *If $p \le q$ and $w(p) \le q$ for some $w \in W_\Pi$, $p, q \in P_2$, then $w = 1$.*

Proof. Suppose that $p = (D, K)$. Then $q = (D_1, K)$ for some D_1 containing D as $p \le q$. But then $w(D) \subseteq D_1$, so $w(D) = D$ as no two distinct subsets D, $w(D)$ of D_1 are conjugate under W_Π, as D_1 is contained in a fundamental domain of W_Π. Hence we can assume that w fixes D as well as K.

The subgroup of W_Π fixing D is a finite reflection group W_D generated by the reflections of W_Π fixing D because $D \subseteq C_\Pi$. The subgroup of W_Π fixing K is generated by the reflections fixing all elements of $K(J)$. Any such reflection is in W_{Ω_K} because R contains 1. However, the condition that $p = (D, K) \in P_2$ implies that there are no reflections of W_{Ω_K} fixing D. Hence the subgroup of W_D fixing K is trivial. So any element $w \in W_\Pi$ such that $w(p) \le q$ is trivial. This proves lemma 3.11.

Lemma 3.12. *The categories Q_2 and Q_3 are equivalent.*

Proof. Lemmas 3.10 and 3.11 show that there is an equivalence of categories from Q_2 to the homotopy quotient of $W_\Pi\backslash P_2$ by $\Gamma_J \times \Gamma_\Pi$. Lemma 3.6 shows that the $\Gamma_J \times \Gamma_\Pi$ posets P_3^+ and $W_\Pi\backslash P_2^+$ are isomorphic. Lemma 3.9 shows that the subset P_2 of P_2^+ corresponds under this isomorphism to the subset $W_\Pi\backslash P_2$ of $W_\Pi\backslash P_2^+$, so the $\Gamma_J \times \Gamma_\Pi$ posets P_3 and $W_\Pi\backslash P_2$ are isomorphic. Therefore the category Q_2 is equivalent to the homotopy quotient of P_3 by $\Gamma_J \times \Gamma_\Pi$, which is just Q_3. This proves lemma 3.12.

Lemma 3.13. *The natural injection from Q_4 to Q_3 is an equivalence of categories.*

Proof. This follows from the fact that Q_4 is a skeleton of Q_3, so the natural injection is an equivalence of categories. This proves lemma 3.13.

We can now prove theorem 2.7. By lemmas 3.3, 3.12, and 3.13, the categories Q_4 and Q_1 are homotopy equivalent. So by lemma 3.1, the component of Q_4 containing $(J, \overline{id_J})$ is a classifying category for Γ_Ω. This proves theorem 2.7.

4. The structure of Γ_Ω.

Theorem 4.1. *The kernel of the natural map from Γ_Ω to $\Gamma_J \times \Gamma_\Pi$ has finite cohomological dimension.*

Proof. The classifying category of the kernel is just a component of the category of the poset P_3. A case by case check on possible Coxeter diagrams shows that the lengths of chains in P_3 are bounded (by $rank(J)+1$ for example), so the corresponding simplicial complex has finite dimension at most $rank(J)$. Therefore the kernel has cohomological dimension at most $rank(J)$. This proves theorem 4.1.

Corollary 4.2. *If Γ_Π has finite virtual cohomological dimension, then so does Γ_Ω.*

Proof. This follows immediately from theorem 4.1 and the fact that Γ_J is finite and standard properties of the virtual cohomological dimension.

Howlett showed that if W_Π is finite then the group Γ_Ω is a subgroup of $Aut(J) \times Aut(\Pi)$. We can deduce this from theorem 4.1 as follows. If W_Π is finite then so is the kernel of the map from Γ_Ω to $Aut(J) \times Aut(\Pi)$. On the other hand this kernel has finite cohomological dimension by theorem 4.1. But any finite group of finite cohomological dimension must be trivial, so the natural map from Γ_Ω to $Aut(J) \times Aut(\Pi)$ is injective. If W_Π is infinite then this kernel is usually infinite, as can be seen from most of the examples below. The fact that this kernel no longer vanishes is the main reason why normalizers of parabolic subgroups of Coxeter groups are more complicated to describe when the Coxeter group is infinite.

5. Examples.

Example 5.1. Suppose that J is A_1 and the group Γ_Π is trivial. In this case Brink [Br] gave an elegant description of the group Γ_Ω as follows. Form the graph obtained from the Coxeter graph Π by keeping only the edges of odd order. Then for any point $J = A_1$ of this new graph, the centralizer of the corresponding reflection (which is the group Γ_Ω corresponding to J) is the fundamental group of this graph with basepoint the chosen point, and in particular Γ_Ω is a free group.

We now check that this is equivalent to the description given by theorem 2.7. The only subdiagrams S with a non R-reflective class $\overline{K}$ are the points of the Coxeter graph or the edges of odd order together with their endpoints. So the category Q_4 has an object for each point or odd order edge of the Coxeter graph. The only non-identity morphisms correspond to inclusions of points in edges. The classifying space of this category is just the first barycentric subdivision of Brink's graph. In particular the fundamental group of this category with some object as basepoint is canonically isomorphic to the fundamental group of Brink's graph with the corresponding point as basepoint. This verifies that the description of Γ_Ω in [Br] is equivalent to the description in theorem 2.7.

The following lemma can often be used to find the fundamental group of a category with at most 2 objects.

Lemma 5.2. *Suppose A and B are subgroups of a group. Let Q be the category with 2 objects p and q such that $Mor(p,p) = A$, $Mor(p,q) = BA$, $Mor(q,q) = B$, $Mor(q,p) = \emptyset$, with composition defined in the obvious way. Then $\pi_1(Q) = A *_{A\cap B} B$.*

Proof. This can be proved by writing down a set of generators and relations for the fundamental group, and checking that they are equivalent to a set of generators and relations for $A *_{A\cap B} B$. We will leave the details to the reader.

For most of the examples below we will take $W_\Pi.\Gamma_\Pi$ to be the group of automorphisms of the even 26 dimensional Lorentzian lattice $II_{1,25}$ not exchanging the two cones of norm 0 vectors. According to Conway [C], the Coxeter group W_Π has a simple reflection $r_\lambda \in \Pi$ for each vector λ of the Leech lattice $\Lambda = \Pi$, and the order of $r_\lambda r_\mu$ is 1, 2, 3, or ∞ according to whether $(\lambda-\mu)^2$ is 0, 4, 6, or greater than 6. The group Γ_Π is the automorphism group $\Lambda.Aut(\Lambda) = \Lambda.(Z/2Z).Co_1$ of the affine Leech lattice, where Λ is the subgroup of translations and Co_1 is Conway's largest sporadic simple group.

If J is a spherical subdiagram of Λ then there is a homomorphism from $N_{W_\Pi.\Gamma_\Pi}(W_J)/W_J$ to the automorphism group of the lattice $J^\perp$. This has finite kernel and co-finite image, so theorem 2.7 can usually be used to describe the automorphism group of the lattice $J^\perp$. Most of the remaining examples in this section use this idea.

Example 5.3. Suppose L is the even Lorentzian lattice of dimension 20 and determinant 3. Let $W^{(2)}(L)$ be the subgroup of $Aut(L)$ generated by reflections of norm -2 vectors of L. Vinberg showed in [V] that $Aut(L)^+/W^{(2)}(L)$ was the automorphism group of a certain K3 surface modulo a cyclic subgroup, and also showed that this group was an extension of a group of order 72 by a free product of 12 groups of order 2. We will show how to recover Vinberg's description of $Aut(L)^+/W^{(2)}(L)$ from theorem 2.7.

We take J to be an $E_6 \subset \Lambda$ so that $L = J^\perp$, and take $R = 1 \subset \Gamma_J = Aut(E_6) = Z/2Z$. The category Q_4 contains exactly two objects, corresponding to an E_6 and an E_7 in Λ. For each subdiagram X of Λ we write $G(X)$ for the automorphisms of $\Pi = \Lambda$ mapping X into itself. The morphisms from the E_6 object to itself form a group $G(E_6)$ of order 72. The morphisms from the E_7 object to itself form a group $Z/2Z \times G(E_7)$ of order $2 \times 6 = 12$ (where the $Z/2Z$ comes from the group Γ_J). The morphisms from E_6 to E_7 can be identified with $Z/2Z \times G(E_6)$. By lemma 5.2 the group Γ_Ω is isomorphic to $G(E_6) *_{G(E_7)} (G(E_7) \times Z/2Z)$. If A, B, and C are any groups with $B \subseteq A$ then $A *_B (B \times C)$ is a semidirect product of a normal subgroup isomorphic to the free product of $|A|/|B|$ copies of C and with the quotient by this normal subgroup isomorphic to A. Hence Γ_Ω has a normal subgroup isomorphic to the free product of 12 copies of $Z/2Z$, and the quotient is the group $G(E_6)$ of order 72. This is equivalent to the description of this group given by Vinberg in [V]. In fact we can describe the various parts of Vinberg's description in terms of the Leech lattice as follows: the group of order 2 is the group of automorphisms of E_6, the group of order 72 is the subgroup of $Aut(\Lambda)$ mapping an E_6 into itself, and the number 12 is the number of E_7's of Λ containing an E_6.

The group $Aut(L)^+$ also contains reflections in norm -6 vectors. The quotient by the full reflection group is finite of order 72, isomorphic to $G(E_6)$. In this case the category Q_4 contains just one point. Note that the reflections of norm -6 vectors induce the nontrivial automorphism of E_6. The 12 elements of order 2 in the paragraph above are in fact reflections of norm -6 vectors. See [V] or [B] for more details of this case.

If L is the even Lorentzian lattice of determinant 4 and dimension 20, which is again the Picard lattice of a $K3$ surface, then Vinberg gave a similar description of the automorphism group as an extension $((Z/2Z)*(Z/2Z)*(Z/2Z)*(Z/2Z)*(Z/2Z)).S_5$ of the symmetric group S_5 by the free product of 5 group of order 2. This group can also be calculated using theorem 2.7. The corresponding category Q_4 has 2 objects, corresponding to a D_6 or D_7 in Λ, the group S_5 is the subgroup of $Aut(\Lambda)$ mapping the D_6 to itself, the group $Z/2Z$ is the group of automorphisms of the D_6 diagram, and the number 5 of copies of $Z/2Z$ is the number of D_7's containing a D_6.

Example 5.4 As a more complicated example we will describe the automorphism group of the Picard lattice L of the "next most algebraic K3 surface", in other words L is the 20 dimensional even Lorentzian lattice of determinant 7. We take $\Pi = \Lambda$, $\Gamma_\Pi = \Lambda.Aut(\Lambda)$, $J = A_6$, $R = 1$, $\Gamma_J = Aut(A_6) = Z/2Z$. Then Γ_Ω is the subgroup of elements of $Aut(L)$ fixing a Weyl chamber of the reflection group generated by the reflections of norm -2 vectors. By theorem 2.7 the group Γ_Ω is the fundamental group of the category Q_4. The category Q_4 has exactly 5 objects, corresponding to the 5 orbits of Coxeter diagrams A_6, A_7, D_7, E_7, and D_8 with a non R-reflective isometry from A_6 into them. (The group $\Lambda.Aut(\Lambda)$ acts transitively on the embeddings of any of these Coxeter diagrams into Λ.) Note that for D_8 and E_7 there is only one equivalence class of isometries from $A_6 = J$ into it, while for A_6, A_7, and D_7 there are two classes, which are exchanged by $Aut(J) = Z/2Z$. The category Q_4 looks like this.

$$\begin{array}{ccccccc}
 & & & & A_7(48) & & \\
 & & (672)\nearrow & & & \searrow(192) & \\
E_7(12) & \longleftarrow & A_6(336) & & \longrightarrow & & D_8(16) \\
 & (672) & (672)\searrow & & (2016) & \nearrow(48) & \\
 & & & & D_7(24) & &
\end{array}$$

Here the numbers are the numbers of morphisms between pairs of objects in Q_4.

Example 5.5. Kondo in [K] studied the automorphism group of a generic Jacobian Kummer surface by embedding its Picard lattice L as the orthogonal complement of a certain $J = A_3A_1^6$ in $II_{1,25}$, and used this to describe a generating set for the automorphism group. (Note that the Leech lattice contains more than 1 orbit of subdiagrams of the form $A_3A_1^6$; the one used by Kondo has largest possible stabilizer in $Aut(\Lambda)$.) By results of Nikulin [N] the automorphism group of the K3 surface is the subgroup of $Aut(L)^+/W^{(2)}(L)$ of elements acting on L'/L as ± 1. This is just the group Γ_Ω of theorem 2.7 where we take $\Pi = \Lambda$, $\Gamma_\Pi = \Lambda.Aut(\Lambda)$, J to be Kondo's $A_3A_1^6$, $R = 1$, and $\Gamma_J =$ the subgroup of order 2 of $Aut(J)$ generated by the nontrivial automorphism of the A_3. The category Q_4 is not connected and the component containing J seems quite complicated. Some partial calculations I have done suggest that the automorphism group of the generic Jacobian Kummer surface might be

$$(W.(Z/2Z)^5)*(Z/2Z)*(Z/2Z)*(Z/2Z)*(Z/2Z)*(Z/2Z)*(Z/2Z)$$

where W is a Coxeter group of rank 32+60 generated by the 16 projections, 16 correlations, sixty Cremona transformations and the $(Z/2Z)^5$ is generated by sixteen translations and a switch [K], but I have not proved this rigorously.

It is much easier to work out the group Γ_Ω for R and Γ_J replaced by $Aut(J) = Z/2Z \times S_6$. In this case the group Γ_Ω has a normal subgroup of index $|S_6|$ isomorphic to the quotient of the automorphism group of a generic Jacobian Kummer surface by the Coxeter group generated by reflections in norm -4 vectors. Theorem 2.7 describes the Γ_Ω as the fundamental group of a component of the finite category Q_4. This component has just two elements, corresponding to the Coxeter diagrams $A_3A_1^6$ and $A_5A_1^5$. Using lemma 5.2 and the results in [K] we see that

$$\Gamma_\Omega = ((Z/2Z)^5.S_6) *_{S_5} (S_5 \times Z/2Z)$$

where $(Z/2Z)^5.S_6$ is the subgroup of Γ_Π fixing $A_3A_1^6$ ([K, lemma 4.5]), and $S_5 \times Z/2Z$ is the subgroup of Γ_Π fixing $A_5A_1^5$. In particular Γ_Ω has a normal subgroup which is the free product of 192 groups of order 2, and the quotient by this normal subgroup is $(Z/2Z)^5.S_6$. If we change R to 1 but keep $\Gamma_J = Aut(J)$ then Γ_Ω becomes the group $Aut(L)^+/W^{(2)}(L)$ which appears to be

$$(W.(Z/2Z)^5.S_6) *_{S_5} (S_5 \times (Z/2Z))$$

though I have not proved this rigorously.

For more examples of automorphism groups of Kummer surfaces, corresponding to the cases $J = D_4^2$, D_4A_3, D_4A_2, or A_3^2, see [K-K].

Example 5.6 Suppose that J in theorem 2.7 contains no components of types A_n $(n \geq 1)$ or D_5, and assume that $R = \Gamma_J = Aut(J)$. Then the map from Γ_Ω to $\Gamma_J \times \Gamma_\Pi$ is injective. This follows because a case by case check over all irreducible spherical Coxeter diagrams shows that any isometry from J into a strictly larger spherical Coxeter diagram is R-reflective.

Example 5.7 We show how to explain Vinberg's result [V, V-K] that the reflection group of $I_{1,n}$ has finite index if and only if $n \leq 19$. Following Conway and Sloane [C-S] we write the even sublattice L of $I_{1,n}$ as $D_{25-n}^\perp$ in $II_{1,25}$ for $n \leq 23$, where $D_3 = A_3$ and $D_2 = A_1^2$. In theorem 2.7 we take $\Pi = \Lambda$, $\Gamma_\Pi = \Lambda.Aut(\Lambda)$, $J = D_{25-n}$, $R = \Gamma_J$ a subgroup of order 2 of $Aut(J)$ (which is equal to $Aut(J)$ for $n \neq 21$). Then the quotient of $Aut(L)^+$ by its reflection subgroup is the group Γ_Ω. For $n \leq 19$ the group Γ_Ω is finite by example 5.6. For $n = 20$ this argument breaks down because J is the "exceptional" case D_5 of example 5.6. For $20 \leq n \leq 23$ we can still describe the group Γ_Ω explicitly using theorem 2.7; see [B, theorem 6.6] for details. When $n = 21$ this gives a natural example with $\Gamma_J \neq Aut(J)$.

If we take J to be D_4 and take $R = \Gamma_J$ to be the symmetric group $S_3 = Aut(D_4)$ instead of a group of order 2 then $W_\Omega.\Gamma_\Omega$ is the automorphism group of the even sublattice of $I_{1,21}$, and Γ_Ω is a finite group. See [B, p. 149] for details.

Example 5.8. The groups Γ_Ω have many of the properties of arithmetic groups; for example, they often have finite classifying categories. (It follows easily from [S] that arithmetic groups have this property.) It is natural to ask when they are arithmetic. There seems to be no obvious general way of deciding this. The following argument can often be used to show that Γ_Ω is not arithmetic. First of all a theorem due to Margulis [Ma, page 3] implies that if a group is an arithmetic subgroup of a group of rank at least 2, then all normal subgroups are either in the center or of finite index. Secondly, a theorem of Borel and Serre [B-S, 11.4.4] says that if a group Γ is arithmetic in a Lie group G then $d = r + vcd(\Gamma)$ where d is the dimension of the symmetric space of G and r is the rank of G and $vcd(\Gamma)$ is the virtual cohomological dimension of Γ. Let us use these results to prove that the group Γ_Ω of example 5.3 (the automorphism group of a K3 surface) is not arithmetic in any Lie group G. The group G must have rank 1 by the theorem of Margulis, as Γ_Ω has non abelian free subgroups of finite index and so cannot be an arithmetic subgroup of a group of rank at least 2. Its virtual cohomological dimension is one, so by the theorem of Borel and Serre [B-S, 11.4.4] the symmetric space of G must have dimension $1+1=2$. But any finite subgroup of Γ_Ω must fix a point of this symmetric space, and therefore acts faithfully on the 2 dimensional tangent space of this point. But Γ_Ω has finite subgroups that are too large to have 2 dimensional faithful representations. Hence Γ_Ω is not arithmetic. Note that Γ_Ω has subgroups of finite index that are free and therefore arithmetic.

References.

[B] R. E. Borcherds, Automorphism groups of Lorentzian lattices, J.Alg. Vol 111 (1987) No. 1, pp. 133-153.

[B-S] A. Borel, J.-P. Serre, Corners and arithmetic groups. Comment. Math. Helv. 48 (1973), 436–491.

[B-H] B. Brink, R. B. Howlett, Normalizers of Parabolic Subgroups in Coxeter Groups, Invent. Math. 136 (1999), no. 2, 323–351.

[Bo] N. Bourbaki, Groupes et algèbres de Lie. Chapitres IV-VI. Actualités Scientifiques et Industrielles, No. 1337 Hermann, Paris 1968.

[Br] B. Brink, On centralizers of reflections in Coxeter groups. Bull. London Math. Soc. 28 (1996), no. 5, 465–470.

[C] J. H. Conway, The automorphism group of the 26-dimensional even unimodular Lorentzian lattice. J. Algebra 80 (1983), no. 1, 159–163. Reprinted as chapter 27 of "Sphere packings, lattices and groups." Grundlehren der Mathematischen Wissenschaften 290. Springer-Verlag, New York, 1993. ISBN: 0-387-97912-3

[C-S] J. H. Conway, N. J. A. Sloane, Leech roots and Vinberg groups. Proc. Roy. Soc. London Ser. A 384 (1982), no. 1787, 233–258. Reprinted as chapter 28 of "Sphere packings, lattices and groups." Grundlehren der Mathematischen Wissenschaften 290. Springer-Verlag, New York, 1993. ISBN: 0-387-97912-3

[D] V. V. Deodhar, On the root system of a Coxeter group, Comm. Algebra 10 (1982) 611–630.

[Hi] H. Hiller, Geometry of Coxeter groups. Research Notes in Mathematics, 54. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982. ISBN: 0-273-08517-4

[H] R. B. Howlett, Normalizers of parabolic subgroups of reflection groups. J. London Math. Soc. (2) 21 (1980), no. 1, 62–80.

[Ke] J. H. Keum, Automorphisms of Jacobian Kummer surfaces. Compositio Math. 107 (1997), no. 3, 269–288.

[K-K] J. H. Keum, S. Kondo, The automorphism groups of Kummer surfaces associated with the product of two elliptic curves, 1998 preprint.

[K] S. Kondo, The automorphism group of a generic Jacobian Kummer surface. J. Algebraic Geometry 7 (1998) 589–609.

[K98] S. Kondo, Niemeier lattices, Mathieu groups and finite groups of symplectic automorphisms of K3 surfaces. Duke Math. Journal Vol 92 (1998) no. 3, 593-598.

[M] S. MacLane, Categories for the working mathematician. Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971.

[Ma] G. A. Margulis, "Discrete subgroups of semisimple Lie groups". Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 17. Springer-Verlag, Berlin, 1991. ISBN: 3-540-12179-X.

[Mu] S. Mukai, Finite groups of automorphisms of K3 surfaces and the Mathieu group. Invent. Math. 94 (1988), no. 1, 183–221.

[N] V. V. Nikulin, An analogue of the Torelli theorem for Kummer surfaces of Jacobians. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 22–41. Integral symmetric bilinear forms and its applications, Math. USSR Izv., 14 (1980), 103-167.

[P-S] I. R. Shafarevich, I. I. Piatetski-Shapiro, Torelli's theorem for algebraic surfaces of type K3. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 35 1971 530–572. English translation in pages 516-557 of Collected mathematical papers by Igor R. Shafarevich. Springer-Verlag, Berlin-New York, 1989. ISBN: 3-540-13618-5

[Q] D. Quillen, Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[S] J.-P. Serre, Cohomologie des groupes discrets. Prospects in mathematics (Proc. Symp., Princeton Univ., Princeton, N.J., 1970), pp. 77–169. Ann. of Math. Studies, No. 70, Princeton Univ. Press, Princeton, N.J., 1971.

[V] È. B. Vinberg, The two most algebraic $K3$ surfaces. Math. Ann. 265 (1983), no. 1, 1–21.

[V-K] È. B. Vinberg, I. M. Kaplinskaja, The groups $O_{18,1}(Z)$ and $O_{19,1}(Z)$. (Russian) Dokl. Akad. Nauk SSSR 238 (1978), no. 6, 1273–1275.