Central extensions of generalized Kac-Moody algebras.

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The main result of Borcherds [1] states that graded Lie algebras with an "almost positive definite" contravariant bilinear form are essentially the same as central extensions of generalized Kac-Moody algebras. In this paper we calculate these central extensions. Ordinary Kac-Moody algebras have nontrivial centers when the Cartan matrix is singular; generalized Kac-Moody algebras turn out to have some "extra" center in their universal central extensions whenever they have simple roots of multiplicity greater than 1, and in particular the dimension of the Cartan subalgebra can be larger than the number of rows of the Cartan matrix.

1. Statement of results.

The main result of this paper states that, roughly speaking, graded Lie algebras with an "almost positive definite" contravariant bilinear form are the same as a sort of generalization of Kac-Moody algebras. More precisely,

Theorem 1. *Suppose that G is a real Lie algebra. Then condition (2) implies (1), and (1) implies (2) if $\dim G_i < \infty$ for all i.*

(1) *G has a grading $G = \oplus_i G_i$ with $G_0 \subset [G, G]$, an involution ω which maps G_i to G_{-i} and is -1 on G_0, and an invariant bilinear form $(,)$ such that $(a, b) = 0$ if a and b have different degrees and such that $(a, -\omega(a)) > 0$ if a is a nonzero element of G_i for $i \neq 0$. In addition G is a sum of one dimensional eigenspaces of G_0; this follows from the other conditions if the spaces G_i, $i \neq 0$, are finite dimensional. We can summarize these conditions roughly by saying that G is a graded Lie algebra with an almost positive definite contravariant bilinear form.*

(2) *There is a symmetric matrix a_{ij}, $i, j \in I$, such that $a_{ij} \leq 0$ if $i \neq j$, and such that if $a_{ii} > 0$ then $2a_{ij}/a_{ii}$ is an integer for any j. Some central extension of G is given by the following generators and relations.*

Generators:

Elements e_i, f_i, h_{ij} for $i, j \in I$.

Relations:

$[e_i, f_j] = h_{ij}$.

$[h_{ij}, e_k] = \delta_i^j a_{ik} e_k$, $[h_{ij}, f_k] = -\delta_i^j a_{ik} f_k$.

If $a_{ii} > 0$ then $\mathrm{Ad}(e_i)^n e_j = 0 = \mathrm{Ad}(f_i)^n f_j$, where $n = 1 - 2a_{ij}/a_{ii}$.

If $a_{ii} \leq 0$, $a_{jj} \leq 0$ and $a_{ij} = 0$ then $[e_i, e_j] = 0 = [f_i, f_j]$.

The subalgebra G_0 must be abelian because the automorphism ω acts as -1 on it. If $a_{ii} > 0$ for all i then the conditions of (2) are equivalent to the defining relations for the Kac-Moody algebra with symmetrized Cartan matrix a (with $h_{ii} = h_i$). If we allow a_{ii} to be non-positive but add the condition that $h_{ij} = 0$ if $i \neq j$ then we obtain the relations for "generalized Kac-Moody algebras" as in Borcherds [1]. In any case, h_{ij} is 0 unless the ith

and jth columns of a are equal, and is in the center of G unless $i = j$, so the Lie algebra above is a central extension of a generalized Kac-Moody algebra.

Theorem 3.1 of Borcherds [1] implies that any Lie algebra satisfying the conditions of (1) above is a central extension of a generalized Kac-Moody algebra. Because of this, Theorem 1 follows almost immediately from the following theorem, which we prove in Section 2 of this paper.

Theorem 2. *Suppose that G is the Lie algebra defined in part (2) of Theorem 1. The subalgebra of G generated by the elements h_{ij} is abelian and has a basis consisting of the elements h_{ij} for $i, j \in I$ such that the ith and jth columns of A are the same. If a has no zero columns then G is perfect and equal to its own universal central extension.*

In particular we recover the well-known result that ordinary Kac-Moody algebras are their own universal central extensions, because in this case it is not possible for two columns of the symmetrized Cartan matrix to be equal.

It is easy to extend most of the results of Borcherds [1] about generalized Kac-Moody algebras, such as the generalized Kac-Weyl formula for characters of highest weight modules, to these central extensions of generalized Kac-Moody algebras.

Example. The monstrous generalized Kac-Moody algebra G [2] has as root lattice the even 26 dimensional unimodular Lorentzian lattice $L = II_{25,1}$, and if r is a nonzero element of L then r has multiplicity $p_{24}(1 - r^2/2)$, which is the number of partitions of $1 - r^2/2$ into parts of 24 colors. This lattice has a certain norm 0 "Weyl vector" ρ, and the simple roots of G are given by

(1) All norm 2 vectors of L which have inner product -1 with ρ. These are the real simple roots of G.

(2) All positive integral multiples of ρ, each with multiplicity 24. These are the norm 0 simple roots of G.

Therefore the universal central extension $\hat{G}$ of G has a Cartan subalgebra which is the sum of

(1) A one dimensional space for each real simple root of G.

(2) A $24^2 = 576$ dimensional space for each positive integer.

The center of $\hat{G}$ has index 26 in the Cartan subalgebra, and the quotient of $\hat{G}$ by its center is simple.

Remark. It is possible to define generalized Kac-Moody superalgebras, and many theorems about generalized Kac-Moody algebras can be generalized to generalized Kac-Moody superalgebras, provided we make a few changes such as replacing the involution ω by an element of order 4 whose square is 1 on even elements of the superalgebra and -1 on odd elements. Theorem 1 generalizes to superalgebras provided we make all the "usual" changes to get from algebras to superalgebras, and also add to part (2) of Theorem 1 the condition that G has no odd real simple roots. There are many finite dimensional simple superalgebras satisfying this condition.

2. Proofs.

In this section we give the proof of Theorem 2, which consists mainly of checking that the Lie algebra in (2) of Theorem 1 is its own universal central extension if it is perfect.

We start by showing that a simpler Lie algebra is its own universal central extension.

Theorem 3. *Let a_{ij} be any real matrix with no zero columns and let G be the Lie algebra given by the following generators and relations.*

Generators:

e_i, f_i, h_{ij}, $i, j \in I$.

Relations:

$[e_i, f_j] = h_{ij}$,

$[h_{ij} e_k] = \delta_i^j a_{ik} e_k$, $[h_{ij} f_k] = -\delta_i^j a_{ik} f_k$.

Then G is perfect and is equal to its own universal central extension.

Proof. It is obvious that G is perfect and that h_{ij} is in the center of G if $i \neq j$. Also,

$$a_{ij} h_{jk} = [a_{ij} e_j, f_k] = [[h_{ii}, e_j], f_k] = [h_{ii}, [e_j, f_k]] - [e_j, [h_{ii}, f_k]]$$
$$= [h_{ii}, h_{jk}] + [e_j, a_{ik} f_k] = [h_{ii}, h_{jk}] + a_{ik} h_{jk}.$$

If $j = k$ this implies that $[h_{ii}, h_{jj}] = 0$, so all the h's commute with each other. If $j \neq k$ this then implies that $h_{jk} = 0$ unless the jth and kth columns of a are equal.

Let e_i', f_j', h_{ij}' be elements of the universal central extension $\hat{G}$ of G mapping onto e_i, f_j, h_{ij}. We repeatedly use the fact that if the images of two elements x' and y' commute in G, then $[x', y']$ is in the center of $\hat{G}$. Then $[h_{ii}', [h_{jj}' e_k']] = [h_{jj}', [h_{ii}', e_k']]$ because $[h_{ii}', h_{jj}']$ is in the center of $\hat{G}$. Therefore

$$a_{jk}[h_{ii}', e_k'] = [h_{ii}', [h_{jj}', e_k']] = [h_{jj}', [h_{ii}', e_k']] = a_{ik}[h_{jj}', e_k'].$$

For any i with a_{ik} nonzero we can form the element $[h_{ii}', e_k']/a_{ik}$ of $\hat{G}$; this does not depend on i by the equality above and maps onto e_k in G. We may therefore assume that e_k' is equal to this element so that $a_{ik} e_k' = [h_{ii}', e_k']$ for all i and may likewise assume that $a_{ik} f_k' = -[h_{ii}', f_k']$. Finally we may redefine $h_{ij}' = [e_i', f_j']$. We wish to show that these new elements satisfy the relations of G, and the only relations which are not part of their definition are that the elements h_{jl}' commute with e_k' and f_k' if $j \neq l$. If we choose i with $a_{ik} \neq 0$ then this follows from

$$a_{ik}[h_{jl}', e_k'] = [h_{jl}', [h_{ii}', e_k']] = [h_{ii}', [h_{jl}', e_k']] = 0,$$

the last equality holding because $[h_{jl}', e_k']$ is in the center of $\hat{G}$. This shows that $\hat{G}$ contains elements satisfying the relations of G, so that the extension $\hat{G}$ splits. Therefore G is its own universal central extension and Theorem 3 is proved.

Remark. The Lie algebra in Theorem 3 is the universal central extension of a Lie algebra associated to the matrix a in Kac [3], and is equal to this Lie algebra when all columns of a are different, which is always the case for ordinary Kac-Moody algebras.

Corollary. *Let G be the Lie algebra of Theorem 3 and let H be any ideal of G. The the universal central extension of G/H is $G/[G, H]$, and in particular if $H = [G, H]$ then G/H is its own universal central extension.*

Proof. This is true for any perfect Lie algebra G which is its own universal central extension. The pullback of $G \rightarrow G/H$ and the universal central extension of G/H is a

perfect central extension of G and hence equal to G, so the map from G to G/H factors through the universal central extension of G/H. It is easy to see that this implies that the universal central extension of G/H is $G/[G, H]$. This proves the corollary.

We can now give the proof of the second part of Theorem 2. If G is the Lie algebra of Theorem 1 and H is its ideal generated by some elements of the form $\mathrm{Ad}(e_i)^n e_j$, $\mathrm{Ad}(f_i)^n f_j$, $[e_i, e_j]$, and $[f_i, f_j]$ as in Theorem 1, then by the corollary we only have to check that $H = [G, H]$. The element $\mathrm{Ad}(e_i)^n(e_j)$ is in $[G, H]$ because $[h_{ii}, \mathrm{Ad}(e_i)^n e_j] = (na_{ii} - a_{ij})\mathrm{Ad}(e_i)^n e_j$ and $na_{ii} - a_{ij}$ is nonzero because $a_{ii} > 0$, $a_{ij} \leq 0$. The element $[e_i, e_j]$ is in $[G, H]$ when $a_{ii} \leq 0$, $a_{jj} \leq 0$, $a_{ij} = 0$ because if we choose any k with $a_{ki} \neq 0$ then $[h_{kk}, [e_i, e_j]] = (a_{ki} + a_{kj})[e_i, e_j]$ and $a_{ki} + a_{kj}$ is nonzero because $a_{ki} < 0$, $a_{kj} \leq 0$. Similarly the elements of the forms $\mathrm{Ad}(f_i)^n f_j$ and $[f_i, f_j]$ are in $[G, H]$, and this proves the second part of Theorem 2.

Finally we have to prove the first part of Theorem 2, the only nontrivial part of which is to prove that the h's are linearly independent. In particular we show that the elements h_{ij} are nonzero if the ith and jth columns of a are the same, so the central extensions we have constructed are nontrivial.

Theorem 4. *Let a be any matrix (possibly with some zero columns) and let G be the Lie algebra with the generators and relations of Theorem 3. Then the subalgebra of G generated by the h's is abelian and has a basis consisting of the elements h_{ij} for $i, j \in I$ such that the ith and jth columns of a are equal. If these columns are not equal then $h_{ij} = 0$.*

Proof. This proof is a slight modification of the one in Kac [3] for the case of Kac-Moody algebras. The only nontrivial thing to check is that the h's are linearly independent, which we show by constructing sufficiently many "lowest weight" representations of G. We let the space V be the universal associative algebra generated by elements e_i, and let e_i act on V by left multiplication. We choose any real numbers b_{ij} with $b_{ij} = 0$ unless the ith and jth columns of a are equal, and define operators h_{ij} on V such that $h_{ij}(1) = b_{ij}1$ and

$$[h_{ij}, e_k] = \delta_i^j a_{ik} e_k \qquad \text{for all } i, j, k. \tag{1}$$

Similarly we can define operators f_j on V such that $f_j(1) = 0$ and

$$[f_j, e_i] = -h_{ij} \qquad \text{for all } i, j, k. \tag{2}$$

This will give a representation of G provided that the relation $[h_{ij} f_k] = -\delta_i^k a_{ik} f_k$ is satisfied, and if it is this will prove the theorem because we will have constructed enough representations of G on which elements of the Cartan subalgebra act non-trivially. The operator $[h_{ij}, h_{kl}]$ commutes with all the e's and vanishes on 1, so it is 0 on V, and for the same reason h_{ij} is 0 when the ith and jth columns of a are different because $b_{ij} = 0$. From the relations (1) and (2) we find that

$$[[h_{ij}, f_k] + \delta_i^j a_{ik} f_k, e_l] = [h_{ij}, [f_k, e_l]] - [f_k, [h_{ij}, e_l]] - \delta_i^j a_{ik} h_{lk}$$
$$= -[h_{ij}, h_{lk}] + \delta_i^j a_{il} h_{lk} - \delta_i^j a_{ik} h_{lk},$$

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which is 0 because h_{lk} is 0 unless $a_{ik} = a_{il}$. Hence $[h_{ij}, f_k] + \delta_i^j a_{ik} f_k$ is 0 on V because it vanishes on 1 and commutes with the e's. This proves Theorem 4.

Theorem 2 follows from this in the same way that the corresponding theorem is proved for Kac-Moody algebras, as in Kac [3].

Remark. If the ith and jth columns of a are equal then the Lie algebra G of Theorem 1 has an outer derivation h defined by $[h, e_i] = e_j$, $[h, e_j] = -e_i$, $[h, e_k] = 0$ if $k \neq i, j$, $[h, f_i] = f_j$, $[h, f_j] = -f_i$, $[h, f_k] = 0$ if $k \neq i, j$. If a has n equal columns then these derivations generate an action of the orthogonal group $O_n(R)$ on G. These outer derivations do not always commute with the elements of the Cartan subalgebra.

Errata. There is a mistake in the statement of Theorem 5.1 of Borcherds [1]: the phrase "nonsingular SCM" should twice be replaced by "nonsingular bilinear form on the Cartan subalgebra". Line 15 on page 502 should read "follows from (4)." Line -4 on page 502 should read "... with all real simple roots." Proposition 2.2 should read "A positive root...".

References.

1. R. E. Borcherds, Generalized Kac-Moody algebras, J. Algebra 115 (1988), 501–512.
2. R. E. Borcherds, The monster Lie algebra, Adv. Math. 83 (1990), 30–47.
3. V. G. Kac, "Infinite dimensional Lie algebras", Birkhäuser, Basel, 1983.