A Monster Lie Algebra?

Chapter 30 of "Sphere packing, lattices and groups" by Conway and Sloane, and Adv. in Math. 53 (1984), no. 1, 75–79.

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We define a remarkable Lie algebra of infinite dimension, and conjecture that it may be related to the Fischer-Griess Monster group.

The idea was mooted in [C-N] that there might be an infinite-dimensional Lie algebra (or superalgebra) L that in some sense "explains" the Fischer-Griess 'Monster" group M. In this chapter we produce some candidates for L based on properties of the Leech lattice described in [C-S]. These candidates are described in terms of a particular Lie algebra L_∞ of infinite rank.

We first review some of our present knowledge about these matters. It was proved by character calculations in [C-N, p. 317] the centralizer C of an involution of class 2A in the Monster group has a natural sequence of modules affording the head characters (restricted to C). In [K], V. Kac has explicitly constructed these as C-modules. Now that Atkin, Fong and Smith [F], [S] have verified the relevant numerical conjectures of [C-N] for M, we know that these modules can be given the structure of M-modules. More recently, Frenkel, Lepowsky, and Meurman [F-L-M] have given a simple construction for the monster along these lines, but this sheds little light on the conjectures.

Some of the conjectures of [C-N] have analogs in which M is replaced by a compact simple Lie group, and in particular by the Lie group E_8. Most of the resulting statements have now been established by Kac and others. However, it seems that this analogy with Lie groups may not be as close as one would wish, since two of the four conjugacy classes of elements of order 3 in E_8 were shown in [Q] to yield examples of modular functions neither of which are the Hauptmodul for any modular group. This disproves the conjecture made on p. 267 of [K], and is particularly distressing since it was the Hauptmodul property that prompted the discovery of the conjectures in [C-N], and it is this property that gives those conjectures almost all their predictive power.

The properties of the Leech lattice that we shall use stem mostly from the facts about "deep holes" in that lattice reported in [C-S Chapter 23]. Let $w = (0, 1, 2, 3, \ldots, 24 \mid 70)$. The main result of [C-S Chapter 26] is that the subset of vectors r in $II_{25,1}$ for which $r \cdot r = 2$, $r \cdot w = -1$ (the "Leech roots') is isometric to the Leech lattice, under the metric defined by $d(r, s)^2 = \text{norm}(r - s)$. The main result of [C-S Chapter 27] is that $\text{Aut}(\text{II}_{25,1})$ is obtained by extending the Coxeter subgroup generated by the reflections in these Leech roots by its group of graph automorphisms together with the central inversion -1. It is remarkable that the walls of the fundamental region for this Coxeter group (which correspond one-for-one with the Leech roots) are transitively permuted by the graph automorphisms, which form an infinite group abstractly isomorphic to the group of all automorphisms of the Leech lattice, including translations.

Vinberg [V] shows that for the earlier analogs $II_{9,1}$ and $II_{17,1}$ of $II_{25,1}$ the fundamental regions for the reflection subgroups have respectively 10 and 19 walls, and the graph automorphism groups have orders 1 and 2. For the later analogs $II_{33,1}, \ldots$, there is no "Weyl vector" like w, so it appears that $II_{25,1}$ is very much a unique object.

We can use the vector w to define a root system in $II_{25,1}$. If $v \in II_{25,1}$ then we define the height of v by $-v \cdot w$, and we say that v is positive or negative according as its height is positive or negative. We now define a Kac-Moody Lie algebra L_∞, of infinite dimension and rank, as follows: L_∞ has three generators $e(r)$, $f(r)$, $h(r)$ for each Leech root r, and is presented by the following relations:

$$\begin{aligned}
[e(r), h(s)] &= r \cdot s \; e(r) \;, \\
[f(r), h(s)] &= -r \cdot s \; f(r), \\
[e(r), f(r)] &= h(r), \\
[e(r), f(s)] &= 0, \\
[h(r), h(r)] &= 0 \;=\; [h(r), h(s)], \\
e(r) \; \{\mathrm{ad} \; e(s)\}^{1-r\cdot s} &= 0 \;=\; f(r) \; \{\mathrm{ad} \; f(s)\}^{1-r\cdot s},
\end{aligned}$$

where r and s are distinct Leech roots. (We have quoted these relations from Moody's excellent survey article [M]. Moody supposes that the number of fundamental roots is finite, but since no argument ever refers to infinitely many fundamental roots at once, this clearly does not matter. See also [K89].)

Then we conjecture that L_∞ provides a natural setting for the Monster, and more specifically that the Monster can be regarded as a subquotient of the automorphism group of some naturally determined subquotient algebra of L_∞.

The main problem is to "cut L_∞ down to size'. Here are some suggestions. A rather trivial remark is that we can replace the Cartan subalgebra H of L_∞ by the homomorphic image obtained by adding the relations

$$c_1 \; h(r_1) \;+\; c_2 \; h(r_2) \;+\; \ldots \;=\; 0$$

for Leech roots $r_1, r_2, \ldots$, whenever $c_1, c_2, \ldots$ are integers for which

$$c_1 r_1 \;+\; c_2 r_2 \;+\; \ldots \;=\; 0 \;.$$

A more significant idea is to replace L_∞ by some kind of completion allowing us to form infinite linear combinations of the generators, and then restrict to the subalgebra fixed by all the graph automorphisms. The resulting algebra, supposing it can be defined, would almost certainly not have any notion of root system.

Other subalgebras of L_∞ are associated with the holes in the Leech lattice, which are either "deep" holes or "shallow" holes (see [C-S Chapter 23]).

(i) By [C-S Chapter 23], any deep hole corresponds to a Niemeier lattice N, which has a Witt part which is a direct sum of root lattices chosen from the list A_n ($n = 1, 2, \ldots$), D_n ($n = 4, 5, \ldots$), E_6, E_7 and E_8. Only 23 particular combinations arise, and we shall take $A_{11} D_7 E_6$ as our standard example. The graph of Leech roots contains a finite subgraph which is the disjoint union of extended Dynkin diagrams corresponding to these Witt components W of N, and so our algebra L_∞ has a subalgebra $L[N]$ which is a direct sum of the Euclidean Lie algebras $E(W)$ corresponding to those components (see [K89]). For example, L_∞ has a subalgebra

$$E(A_{11}) \ + \ E(D_7) \ + \ E(E_6) \ .$$

Each such subalgebra of L_∞ can be extended to a larger subalgebra $L^*(N)$ having one more fundamental root, corresponding to a "glue vector' of the appropriate hole (see [C-S Chapter 25]). In the corresponding graph, the new node is joined to a single special node in each component. The graph for $L^*[A_{11}D_7E_6]$ is shown in Figure 30.1 of [C-S]. (A special node of a connected extended Dynkin diagram is one whose deletion would result in the corresponding ordinary diagram.) These hyperbolic algebras $L^*[N]$, having finite rank, are certainly more manageable than L_∞ itself. Since the 23 Niemeier lattices yield 23 constructions for the Leech lattice ([C-S Chapter 25]), it is natural to ask if we can obtain 23 different constructions for the Monster using the Lie algebras $L[N]$ or $L^*[N]$.

(ii) Each shallow hole in the Leech lattice (see [C-S Chapter 24]) corresponds to a maximal subalgebra of L_∞ of finite rank.

We are making various calculations concerning L_∞ (finding the multiplicities of certain roots via the Weyl-MacDonald-Kac formula, etc.). It is worth noting that these calculations are facilitated by the remarkable recent discovery that the Mathieu group M_{12} is generated by the two permutations

$$t \ \mapsto \ |2t| \ , \qquad t \ \mapsto \ 11 - t \ (\mathrm{mod}\ 23) \ ,$$

of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, where $|x|$ denotes the unique y in this set for which $y \ \equiv \ \pm x$ (mod 23). (See [C-S Chapter 11]. This discovery arose from the study of properties of various standard playing-card shuffles. We have noticed that there are many other elements of M_{12} which have simple formulae in this "card numbering", for example $t \ \mapsto \ | \ t^3 \ |$.) The simplest transformation (see [C-S Chapter 28]) between the usual Euclidean coordinates for the Leech lattice and its Lorentzian coordinates uses this description of M_{12}.

References.

[C-N] J. H. Conway, S. Norton, Monstrous moonshine, Bull. London. Math. Soc. 11 (1979) 308-339.

[C-S] J. H. Conway, N. J. A. Sloane, Sphere packings, lattices and groups. Third edition. Grundlehren der Mathematischen Wissenschaften 290. Springer-Verlag, New York, 1999. ISBN: 0-387-98585-9

[F] P. Fong, Characters arising in the Monster–modular connection. P.S.P.M. 37 (1980), 557–559.

[F-L-M] I. B. Frenkel, J. Lepowsky, A. Meurman, Vertex operator algebras and the monster, Academic press 1988.

[K] V. G. Kac, An elucidation of "Infinite dimensional algebras ... and the very strange formula". Adv. in Math. 35 (1980), 264–273.

[K89] V. G. Kac, "Infinite dimensional Lie algebras", third edition, Cambridge University Press, 1990.

[M] R. V. Moody, Root systems of hyperbolic type, Adv. in Math. 33 (1979), 144–160.

[Q] L. Queen, Some relations between finite groups, Lie groups, and modular functions, PhD. dissertation, Univ. of Cambridge, 1980. Modular functions and finite simple groups, PSPM 37 (1980), 561–566. Modular functions arising from some finite groups, MTAC 37 (1981), 547–580.

[S] R. F. Smith, On the head characters of the monster simple group, In "Finite groups–coming of age", edited by J. McKay, Contemp. Math. 45 (1985) p. 303–313.

[V] È. B. Vinberg, Some arithmetical discrete groups in Lobačevskiĭ spaces. Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), pp. 323–348. Oxford Univ. Press, Bombay, 1975.

Remark added 1998: The Lie algebra of this paper is indeed closely related to the monster simple group. In order to get a well behaved Lie algebra it turns out to be necessary to add some imaginary simple roots to the "Leech roots". This gives the *fake* monster Lie algebra, which contains the Lie algebra of this paper as a large subalgebra. See "The monster Lie algebra", Adv. Math. Vol. 83, No. 1, Sept. 1990, for details (but note that the fake monster Lie algebra is called the monster Lie algebra in this paper). The term "monster Lie algebra" is now used to refer to a certain "$Z/2Z$-twisted" version of the fake monster Lie algebra. The monster Lie algebra is acted on by the monster simple group, and can be used to show that the monster module constructed by Frenkel, Lepowsky, and Meurman satisfies the moonshine conjectures; see "Monstrous moonshine and monstrous Lie superalgebras", Invent. Math. 109, 405-444 (1992).