4 The exponential map

If A is a matrix, we can define $\exp(A)$ by the usual power series. We should check this converges: this follows if we define the norm of a matrix to be $\sup_{x\neq 0}(|Ax|)/|x|$. Then $|AB| \le |A||B|$ and $|A+B| \le |A|+|B|$ so the usual estimates show that the exponential series of a matrix converges. The exponential is a map from the Lie algebra $M_n(R)$ of the Lie group $GL_n(R)$ to $GL_n(R)$. (The same proof shows that the exponential map converges for bounded operators on a Banach space. The exponential map also exists for unbounded self-adjoint operators on a Hilbert space, but this is harder to prove and uses the spectral theorem.) The exponential map satisfies $\exp(A+B) = \exp(A)\exp(B)$ whenever A and B commute (same proof as for reals) but this does NOT usually hold if A and B do not commute. Another useful identity is $\det(\exp(A)) = \exp(\mathrm{trace}(A))$ (conjugate A to an upper triangular matrix).

To calculate the exponential of a matrix explicitly one can use the Lagrange interpolation formula as in the following exercises.

Exercise 49 Show that if the numbers λ_i are n distinct numbers, and B_i are numbers, then

$$B_1\frac{(A-\lambda_2)(A-\lambda_3)\cdots}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots}+B_2\frac{(A-\lambda_1)(A-\lambda_3)\cdots}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots}+\cdots$$

is a polynomial of degree less than n taking values B_i at λ_i.

Exercise 50 Show that if the matrix A has distinct eigenvalues λ_1,λ_2,... then $\exp(A)$ is given by

$$\exp(\lambda_1)\frac{(A-\lambda_2)(A-\lambda_3)\cdots}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots}+\exp(\lambda_2)\frac{(A-\lambda_1)(A-\lambda_3)\cdots}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots}+\cdots$$

(In this formula exp can be replaced by any holomorphic function.)

Exercise 51 Find $\exp\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The work can be reduced a little by writing the matrix as a sum of a multiple of the identity and a matrix of trace 0.

In particular for every element of the Lie algebra we get a 1-parameter subgroup $\exp(tA)$ of the Lie group. We look at some examples of 1-parameter subgroups.

Example 52 If A is nilpotent, then $\exp(tA)$ is a copy of the real line, and its elements consist of unipotent matrices. In this case the exponential series is just a polynomial, as is its inverse $\log(1+x)$, so the exponential map is an isomorphism between nilpotent matrices and unipotent ones.

Example 53 If the matrix A is semisimple with all eigenvalues real, then it can be diagonalized, and the image of the exponential map is a copy of the positive real numbers. In particular it is again injective.

Example 54 If the matrix A is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (semisimple with imaginary eigenvalues) then the image of the exponential map is the circle group of rotations. In particular the exponential map is no longer injective.

Example 55 A 1-parameter subgroup need not have closed image: consider an irrational line in the torus $S^1 \times S^1$, considered as (say) diagonal matrices in $GL_2(\mathbb{C})$.)

In general a 1-parameter subgroup may combine features of all the examples above.

Exercise 56 Show that if A is in the Lie algebra of the orthogonal group (so $A + A^t = 0$) then $\exp(A)$ is in the orthogonal group.

One way to construct a Lie group from a Lie algebra is to fix a representation of the Lie algebra on a vector space V, and define the Lie group to be the group generated by the elements $\exp(a)$ for a in the Lie algebra. It is useful to do this over fields other than the real numbers; for example, we might want to do it over finite fields to construct the finite simple groups of Lie type. The problem is that the exponential series does not seem to make sense. We can get around this in two steps as follows. First of all, if we work over (say) the rational numbers, the exponential series still makes sense on nilpotent elements of the Lie algebra, as the series is then just a finite polynomial. The other problem is that the exponential series contains coefficients of $1/n!$, that make no sense if $n \leq p$ for p the characteristic of the field. Chevalley solved this problem as follows. The elements $a^n/n!$ are elements of the universal enveloping algebra over the rationals. If we take the universal enveloping algebra over the integers and reduce it mod p we cannot then divide a^n by $n!$. However we can first do the division by $n!$ asnd then reduce mod p: in other words we take the subring of the universal enveoping algebra over the rationals generated by the elements $a^n/n!$ for a nilpotent, and then reduce this subring mod p. Then this has well defined exponential maps for nilpotent elements of the Lie algebra.

Another way to define exponentials without dividing by a prime p is to use the Artin-Hasse exponential

$$\exp\left(\frac{x}{1} + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots\right)$$

Exercise 57 Show that a formal power series $f(x) = 1 + \cdots$ with rational coefficients has coefficients with denominators prime to p if and only if $f(x^p)/f(x)^p \equiv 1 \bmod p$. Use this to show that the Artin-Hasse power series has coefficients with denominators prime to p.

Example 58 The exponential map need not be onto, even if the Lie group is connected. As an example, we will work out the image of the exponential map for the connected group $SL_2(R)$. The Lie algebra is the 2 by 2 matrices of trace 0, so the eigenvalues are of the form $\lambda, -\lambda$ for $\lambda > 0$, or $i\lambda, -i\lambda$ for $\lambda > 0$, or $0, 0$. In the first case $\exp(A)$ is diagonalizable with two positive distinct eigenvalue with product 1. In the second case A is diagonalizable with two eigenvalues of absolute value 1 and product 1. In the third case A is unipotent (both eigenvalues 1) but need not be diagonalizable. If we check through the conjugacy classes of $SL_2(R)$ we see that we are missing the following classes: matrices with two distinct negative eigenvalues (in other words trace less than -2), and non-diagonalizable matrices with both eigenvalues -1. So the image of the exponential map is not even dense (or open or closed): it omits all matrices of trace less than -2.

There is an alternative more abstract definition of the exponential map that goes roughly as follows. For any element a of the Lie algebra of a group G , we show that there is a unique 1-parameter subgroup $\mathbb{R} \mapsto G$ whose derivative at the origin is a. Then $exp(a)$ is defined to be the value of this 1-parameter subgroup at 1. This definition has the advantage that it works for all Lie groups, and in particular shows that the expnential map does not depend on a choice of representation of the Lie group as a matrix group. The disadvantage is that one has to prove existence and uniqueness of 1-parameter subgroups, which are essentially geodesics for a suitbale connection on G.

Theorem 59 *(Campbell-Baker-Hausdorff)*

$$\exp(A)\exp(B) = \exp(A + B + [A, B]/2 + \cdots)$$

where the exponent on the right is an infinite formal series in the free Lie algebra generated by formal variables A and B.

In particular this justifies the claim that Lie algebras capture the local structure of a Lie group, because we can define the product of the Lie group locally in terms of the Lie bracket. The convergence of the Campbell-Baker-Hausdorff formula is a bit subtle: it converges in some neighborhood of 0, and converges for nilpotent Lie algebras, but does not converge everwhere if the simply connected Lie group of the Lie algebra is not homeomorphic to a vector space. For example, if it converged everywhere for the group $SU(2)$ then we would find that R^3 can be given a group structure locally isomorphic to it, which is impossible as the universal cover is not R^3.