The universal enveloping algebra is given by "generators and relations", and when objects are given in this way it is usually easy to find upper bounds for the size of such objects by algebraic manipulation, but harder to show they do not collapse further. A good way to find a lower bound on the size is to find an explicit representation on something. Some cases of the PBW theorem are easy: for example, if our Lie algebra is known to be the Lie algebra of a Lie group, then the UEA should be differential operators, which come with a natural representation on smooth functions, or we can even restrict to formal power series expansions of smooth functions at the identity. There are enough such functions to see that the elements $g_1^{n_1} g_2^{n_2} ...$ of the UEA are linearly independent as they have linearly independent actions on smooth functions.

If we want to prove the PBW theorem more generally we have to work harder: even over the reals it is not at all obvious that any Lie algebra is the Lie algebra of a Lie group. (At first sight this seems easy: all one has to do is find a faithful representation on a vector space, and exponentiate this to a group action. For abelian Lie algebras this is trivial, and at the opposite extreme when the algebra has no center it is also trivial as we can use the adjoint representation. Every Lie algebra can be obtained by starting with these two type and taking extensions, but it the trouble is that it is hard to show that one can find a faithful representation of an extension of two algebras with faithful representations.)

In general we have to build a suitable representation from the Lie algebra. We will do this by building something that "ought" to be Ug, and then defining an action of g on this by left multiplication and checking that it works.

Step 1: Construction of V. We choose a well-ordered basis of g and define V to have a basis of monomials that are formal products $abc...$ of elements of the basis with $a \leq b \leq c \leq \cdots$. (There is an obvious map from V onto Ug; our aim is to prove that it is an isomorphism.)

Step 2: Construction of the action of g on V. Suppose a is a basis element of g and $bc\cdots$ is in V. We define $a(bc\cdots)$ to be $abc\cdots$ if $a \leq b$, and $[ab](c\cdots) + b(a(c\cdots)$ if $a > b$. This is well defined by induction on the length of an element of V and by using the fact that the basis is well ordered.

Step 3. Check that this action of g on V satisfies $[a, b] = ab - ba$. This holds on $c\cdots$ whenever $b \leq c$ by definition of the action, and similarly it holds whenever $a \leq c$, so we can assume that both a and b are greater than c. We can also assume by induction that $[x, y](z) = x(y(z)) - y(x(z)$ for any elements x, y of the Lie algebra and any element z of V of length less than that of $c\cdots$. We calculate both sides of the identity we have to prove, pushing c to the left, as follows:

$$
\begin{aligned}
a(b(c\cdots)) &= a(c(b(\cdots)) + a([b,c](\cdots)) && (1)\\
&= c(a(b(\cdots))) + [a,c](b(\cdots) + [a,[b,c]](\cdots) + [b,c](a(\cdots)) && (2)
\end{aligned}
$$

and similarly

$$b(a(c\cdots)) = c(b(a(\cdots))) + [b,c](a(\cdots) + [b,[a,c]](\cdots) + [a,c](b(\cdots))$$

$$[a,b](c\cdots) = c([a,b](\cdots)) + [[a,b],c](\cdots)$$

Comparing everything we see that

$$a(b(c\cdots)) - b(a(c\cdots)) - [a,b](c\cdots)$$

is equal to

$$[[a,b],c](\cdots)+[[b,c],a](\cdots)+[[c,a],b](\cdots)$$

We finally use the Jacobi identity for g to see that this vanishes, thus showing that we indeed have an action of g on V.

This completes the proof of the PBW theorem.

The PBW theorem shows that we have found "all" identities satisfied by the Lie bracket, at least over fields, because any Lie algebra is a subalgebra of the Lie algebra of some associative algebra. For Jordan algebras the analogous result is not true. Jordan algebras are analogous to Lie algebras except that the Jordan product $a \circ b$ is $ab+ba$ rather than $ab-ba$. This satisfies identities $a \circ b = b \circ a$ and the Jordan identity. However these identities are not enough to force the Jordan algebra to be a subalgebra of the Jordan algebra of an associative ring: the Jordan algebras with this property are called special, and satisfy further independent identities (the smallest of which has degree 8). There is a 27-dimensional Jordan algebra (Hermitian matrices over the Cayley numbers) that is not special. Another example is Lie superalgebras: here we need the extra identity $[a,[a,a]]=0$ for a odd in order to get a faithful representation in a ring.

The PBW theorem underlies the later calculations of characters of irreducible representations of Lie algebras. These representations can be written in terms of "Verma modules", an Verma modules in turn can be identified with the universal enveloping algebras of Lie algebras. The PBW theorem gives complete control over the "size" of the Verma module, in other words its character, which in turn leads to character formulas for the irreducible representations. A related application is the construction of exceptional simple Lie algebras (and Kac–Moody Lie algebras): the hard part of the construction is to show that these algebras are non-zero, which ultimately reduces to showing that certain universal enveloping algebras are non-zero.

The UEA of a Lie algebra is not just an associative algebra; it is also a Hopf algebra.

Definition 36 *A Hopf algebra is a group.*

In order to understand this, we need to explain what a group is. It is a set G with an associative product, identity, and inverse. However it also has further structure as follows. Given a group action on sets X, Y, there is an action on $X \times Y$ given by $g(x \times y) = g(x) \times g(y)$. This action is given by the diagonal map $g \mapsto g \times g$ from G to $G \times G$. Similarly there is an action of G on a 1-point set, induced by a map $g \mapsto *$ from G to a 1-point set. These extra maps are not usually mentioned in the definition of a group, because they are uniquely determined. There is a unique diagonal map from G to $G \times G$ so that the maps $G \mapsto G \times G \mapsto G$ are identities. So we have a coassociative coproduct map $G \mapsto G \times G$ and a counit $G \mapsto 1$ making G into a "cogroup", but this structure is boring because it is uniquely determined for any set. The reason it is unique is that we define the product of a group using the categorical product of sets. Similarly we can define groups in any category with products, and again the "coalgebra" structure is uniquely determined.

Now suppose that we have a category with some sort of product operation that is not the categorical product; for example, the tensor product of modules over a ring. If we copy the naive definition of a group, we get the concept of an associative algebra. However these do not behave like groups: for example, there is not natural action on tensor products of modules, corresponding to an action of a group on a product of sets. To get this we need to put in the "costructure" explicitly. In other words, we need a coassociative product with a counit, so that the coproduct is an automorphism of algebras. The inverse of a group gives an antipode on the Hopf algebra.

So we can defined groups/Hopf algebras in any symmetric monoidal category. In the category of sets, these are just the usual groups, while in the category of modules over a ring we get the usual Hopf algebras.

Example 37 If G is a group, then its group ring $R[G]$ over a commutative ring is a Hopf algebra, with the coproduct given by $g \mapsto g \otimes g$ and the counit given by $g \mapsto 1$.