38 Discrete subgroups of Lie groups

We will show how to classify the finite subgroups G of $O_3(R)$ using Thurston's theory of orbifolds. The key idea it to look at the orbit space S^2/G. This is not usually a manifold (unless G happens to act freely) but is a more general sort of object called an orbifold, which is a sort of manifold with mild singularities. Two-dimensional orbifolds turn out to be easy to classify as there are only a limited number of possible singularities, so we can use this to classify finite rotation and reflection groups.

We first look at the possible singularities of a smooth surface by a finite group. The singularity is going to look locally like R^2/G where G is some finite group acting on the vector space R^2. There are not too many of these, as they have to be subgroups of $O_2(R)$: G is either cyclic, acting as rotations or dihedral, acting as rotations and reflections. The corresponding singularities are either conical points with angle $2\pi/n$ (for G cyclic of order n) or a sector of angle $2\pi/2n$, for G dihedral of order $2n$.

The Euler characteristic of the orbifold is (Euler characteristic of 2-sphere)/(Order of group), which must be positive as the 2-sphere has positive Euler characteristic. This is also equal to the number of points-lines plus faces of the orbifold. However we must be careful to count points and lines properly: a point that is a quotient singularity by a group of order n is really only $1/n$ of a point. Similarly a boundary edge is really only half a line. In other words whenever we have a point that is a quotient singularity we need to adjust the Euler characteristic by $1 - 1/n$.

So in the orientable case when the quotient orbifold is a topological sphere with quotient singularities of orders n_i the Euler characteristic is

$$\text{Euler characteristic of sphere} - \text{corrections} = 2 - \sum(1 - 1/n_i)$$

and this is equal to (Euler characteristic of 2-sphere)/(Order of group) $= 2/|G| > 0$. So we want to solve the equation

$$\sum(1 - 1/n_i) < 2$$

for integers $n_i > 1$. There is an extra condition: there cannot be just one integer, and if we have just two integers they must be the same, otherwise we get a bad orbifold which has no cover that is a manifold. It is easy to find the solutions: they are $()$, (n, n), $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. These correspond to quotients of the sphere by the trivial group, cyclic groups of order n, dihedral groups of order $2n$, and the rotations of a tetrahedron, octahedron, or icosahedron.

The unorientable case when the quotient is a disc or projective plane is similar. The correction factors are $1/2$ for each edge and $1 - 1/2m$ for every sector singularity on the boundary, so we find

$$1 - \sum(1 - 1/n_i) - \sum(1 - 1/m_i)/2 = 1/|G| > 0.$$

Here the numbers m_i are the orders of the sector singularities, and the 2's are replaced by 1's because we replace the sphere of Euler characteristic 2 by the disc or projective plane which have Euler characteristic 1. (Of course for the projective plane there are no sector singularities because it has no boundary.)

The finite subgroups of $O_4(R)$ can be classified by reducing to the case of $O_3(R)$ using the fact the $O_4(R)$ is almost a product of 2 copies of $O_3(R)$: more precisely the double cover of $SO_4(R)$ is isomorphic to the product of two copies of the double cover of $SO_3(R)$. These groups act on S^3 so give examples of compact 3-dimensional orbifolds. If the finite group acts freely on S^3 then we get compact 3-dimensional manifolds called space forms. These are fairly straightforward to classify, and by Perelman's proof of Thurston's elliptization conjecture they give all closed 3-manifolds with finite fundamental group. For example, lens spaces arise by taking a 2-dimensional complex representation of a cyclic group, which acts on the 3-sphere of vectors of length 1. A particularly famous example is Poincare's homology 3-sphere. Any subgroup of the group S^3 automatically acts fixed point freely on S^3, so any double cover of a finite group of rotations on $SO_3(R)$ is the fundamental group of a compact 3-manifold. The Poincare 3-sphere arises by taking the double cover of the group of rotations of the icosahedron. The fundamental group is the perfect group of order 120. As the fundamental group is perfect, the first homology group of the manifold is trivial, so the manifold is a homology sphere.

Exercise 401 Classify the 17 wallpaper groups (discrete co-compact subgroups of the group $R^2.O_2(R)$ of isometries of the plane R^2) by looking at the quotient orbifolds. This is similar to classifying the finite subgroups of isometries of the sphere, except this time the Euler characteristic of the quotient orbifold is zero rather than positive. There are 4 orbifolds whose underlying topological space is a sphere and 8 whose underlying space is a disk. Also there are 5 more surfaces that can appear: the Klein bottle, torus, Moebius strip, projective plane, and annulus, each of which corresponds to 1 group.

In 3-dimensions the analogues of wallpaper groups are the cocompact subgroups of $R^3.O_3(R)$ and are called space groups. They are of great interest to crystallographers and geologists as they describe the possible symmetries of crystals. There are 230 or 219 classes of them depending on whether one distinguishes mirror images of group, which were independently classified by Fyodorov, Barlow, and Schnflies in the 1890s. Conway and Thurston redid the classification in terms of 3-dimensional orbifolds.

A Fuchsian group is a discrete subgroup of $SL_2(R)$. The group $SL_2(R)$ acts on 2-dimensional hyperbolic space and also on the upper half complex plane, so Fuchsian groups are closely related to 2-dimensional oriented hyperbolic manifolds and to Riemann surfaces. (The correspondence is not quite exact, as Fuchsian groups can have elliptic elements fixing points, so that the quotients have orbifold singularities.) In particular any compact Riemann surface of genus g at least 2 gives a Fuchsian group isomorphic to its fundamental group. The space of (marked) Fuchsian groups obtained like this is called Teichmuller space, and is a complex manifold of dimension $3g - 3$ with a very rich geometry. The classical example of a Fuchsian group is $SL_2(Z)$. The quotient of the upper half plane by $SL_2(Z)$ is an orbifold given by the complex sphere with a point missing and with orbifold singularities of orders 2 and 3. The more general case of congruence subgroups such as $\Gamma_0(N)$ (c divisible by N) are of central importance in number theory: Wiles's proof of Fermat's last theorem depended on a deep study of the modular forms for these groups (sections of certain line bundles over the quotient orbifolds).

A Kleinian group is a discrete subgroup of $SL_2(C)$. Like Fuchsian groups, these can be thought of as either complex or hyperbolic transformation groups, because the group $PSL_2(C)$ is both the group of automorphisms of the complex sphere, and also the group of automorphisms of oriented hyperbolic 3-space. In particular if Kleinian group is cocompact and acts freely, we get a closed hyperbolic 3-manifold. Unlike the case of Fuchsian groups, where there are infinite families of cocompact groups, cocompact (and cofinite) Kleinian groups are rigid: the only deformations are given by inner automorphisms. This is a special case of Mostow rigidity, which says in particular that isomorphisms of cofinite Kleinian groups extend to isomorphisms of $PSL_2(C)$. One way of thinking about this is that closed hyperbolic 3-manifolds have an essentially unique hyperbolic metric (quite unlike what happens in 2-dimensions): any invariant of the metric, such as the volume, is an invariant of the underlying topological manifold.

Thurston discovered that all closed 3-manifolds can be built from discrete subgroups of Lie groups. More precisely every closed 3-manifold can be cut up in a canonical way along spheres and tori so that each piece has a geometric structure of finite volume. There are 8 different sorts of geometric structure that appear corresponding to 8 particular Lie groups (modulo a maximal compact subgroup). We have seem 3 of these above: they correspond to spherical, Euclidean, and hyperbolic structures. Two more come from taking the product of 2-dimensional spherical or hyperbolic structures with a line. The remaining 3 structures come from 3 of the Bianchi groups: the Heisenberg group, the universal cover of $SL_2(R)$, and a solvable group that is the Bianchi group of type VI_0 and its identity component is the group of isometries of Minkowski space in 2-dimensions. Of these 8 geometries, the manifolds for all except the hyperbolic geometry are classified (or at least well understood). There seems to be no obvious analogue of Thurston's classification for manifolds of dimension 4 and above.