37 Simple real Lie algebras

37.1 Real forms

(The stuff about E8 is duplicate and needs to be removed)

In general, suppose L is a Lie algebra with complexification $L \otimes \mathbb{C}$. How can we find another Lie algebra M with the same complexification? $L \otimes \mathbb{C}$ has an anti-linear involution $\omega_L : l \otimes z \mapsto l \otimes \bar{z}$. Similarly, it has an anti-linear involution ω_M. Notice that $\omega_L \omega_M$ is a linear involution of $L \otimes \mathbb{C}$. Conversely, if we know this involution, we can reconstruct M from it. Given an involution ω of $L \otimes \mathbb{C}$, we can get M as the fixed points of the map $a \mapsto \omega_L \omega(a)$ "=" $\overline{\omega(a)}$. Another way is to put $L = L^+ \oplus L^-$, which are the $+1$ and -1 eigenspaces, then $M = L^+ \oplus iL^-$.

Thus, to find other real forms, we have to study the involutions of the complexification of L. The exact relation is subtle, but this is a good way to go.

Example 393 Let $L = \mathfrak{sl}_2(\mathbb{R})$. It has an involution $\omega(m) = -m^T$. $\mathfrak{su}_2(\mathbb{R})$ is the set of fixed points of the involution ω times complex conjugation on $\mathfrak{sl}_2(\mathbb{C})$, by definition.

So to construct real forms of E_8, we want some involutions of the Lie algebra E_8 which we constructed. What involutions do we know about? There are two obvious ways to construct involutions:

1. Lift -1 on L to $\hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$, which induces an involution on the Lie algebra.
2. Take $\beta \in L/2L$, and look at the involution $\hat{e}^\alpha \mapsto (-1)^{(\alpha,\beta)}\hat{e}^\alpha$.

(2) gives nothing new: we get the Lie algebra we started with. (1) only gives one real form. To get all real forms, we multiply these two kinds of involutions together.

Recall that $L/2L$ has 3 orbits under the action of the Weyl group, of size 1, 120, and 135. These will correspond to the three real forms of E_8. How do we distinguish different real forms? The answer was found by Cartan: look at the signature of an invariant quadratic form on the Lie algebra.

A bilinear form (,) on a Lie algebra is called *invariant* if $([a,b],c)+(b[a,c]) = 0$ for all a, b, c. This is called invariant because it corresponds to the form being invariant under the corresponding group action. Now we can construct an invariant bilinear form on E_8 as follows:

1. $(\alpha, \beta)_{\text{in the Lie algebra}} = (\alpha, \beta)_{\text{in the lattice}}$
2. $(\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}) = 1$
3. $(a, b) = 0$ if a and b are in root spaces α and β with $\alpha + \beta \neq 0$.

This gives an invariant inner product on E_8, which we prove by case-by-case check

Exercise 394 do these checks

We constructed a Lie algebra of type E_8, which was $L \oplus \bigoplus \hat{e}^\alpha$, where L is the root lattice and $\alpha^2 = 2$. This gives a double cover of the root lattice:

$$1 \to \pm 1 \to \hat{e}^L \to e^L \to 1.$$

We had a lift for $\omega(\alpha) = -\alpha$, given by $\omega(\hat{e}^\alpha) = (-1)^{(\alpha^2/2)}(\hat{e}^\alpha)^{-1}$. So ω becomes an automorphism of order 2 on the Lie algebra. $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$ is also an automorphism of the Lie algebra.

Suppose σ is an automorphism of order 2 of the real Lie algebra $L = L^+ + L^-$ (eigenspaces of σ). We saw that you can construct another real form given by $L^+ + iL^-$. Thus, we have a map from conjugacy classes of automorphisms with $\sigma^2 = 1$ to real forms of L. This is not in general in isomorphism.

E_8 has an invariant symmetric bilinear form $(e^\alpha, (e^\alpha)^{-1}) = 1$, $(\alpha, \beta) = (\beta, \alpha)$. The form is unique up to multiplication by a constant since E_8 is an irreducible representation of E_8. So the *absolute value of the signature* is an invariant of the Lie algebra.

For the split form of E_8, what is the signature of the invariant bilinear form (the split form is the one we just constructed)? On the Cartan subalgebra L, $(\ ,\ )$ is positive definite, so we get $+8$ contribution to the signature. On $\{e^\alpha, (e^\alpha)^{-1}\}$, the form is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so it has signature $0 \cdot 120$. Thus, the signature is 8. So if we find any real form with a different signature, we will have found a new Lie algebra.

We first try involutions $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$. But this does not change the signature. L is still positive definite, and we still have $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ on the other parts. These Lie algebras actually turn out to be isomorphic to what we started with (though we have not shown that they are isomorphic).

Now try $\omega : e^\alpha \mapsto (-1)^{\alpha^2/2}(e^\alpha)^{-1}$, $\alpha \mapsto -\alpha$. What is the signature of the form? We write down the $+$ and $-$ eigenspaces of ω. The $+$ eigenspace will be spanned by $e^\alpha - e^{-\alpha}$, and these vectors have norm -2 and are orthogonal. The $-$ eigenspace will be spanned by $e^\alpha + e^{-\alpha}$ and L, which have norm 2 and are orthogonal, and L is positive definite. What is the Lie algebra corresponding to the involution ω? It will be spanned by $e^\alpha - e^{-\alpha}$ where $\alpha^2 = 2$ (norm -2), and $i(e^\alpha + e^{-\alpha})$ (norm -2), and iL (which is now negative definite). So the bilinear form is *negative definite*, with signature $-248 (\neq \pm 8)$.

With some more work, we can actually show that this is the Lie algebra of the *compact* form of E_8. This is because the automorphism group of E_8 preserves the invariant bilinear form, so it is contained in $O_{0,248}(\mathbb{R})$, which is compact.

Now we look at involutions of the form $e^\alpha \mapsto (-1)^{(\alpha,\beta)}\omega(e^\alpha)$. Notice that ω commutes with $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$. The β's in (α, β) correspond to $L/2L$ modulo the action of the Weyl group $W(E_8)$. Remember this has three orbits, with 1 norm 0 vector, 120 norm 2 vectors, and 135 norm 4 vectors. The norm 0 vector gives us the compact form. Let's look at the other cases and see what we get.

Suppose V has a negative definite symmetric inner product $(\ ,\ )$, and suppose σ is an involution of $V = V_+ \oplus V_-$ (eigenspaces of σ). What is the signature of the invariant inner product on $V_+ \oplus iV_-$? On V_+, it is negative definite, and on iV_- it is positive definite. Thus, the signature is $\dim V_- - \dim V_+ = -\mathrm{tr}(\sigma)$. So we want to work out the traces of these involutions.

Given some $\beta \in L/2L$, what is $\mathrm{tr}(e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha)$? If $\beta = 0$, the traces is obviously 248 because we just have the identity map. If $\beta^2 = 2$, we need to figure how many roots have a given inner product with β. Recall that this was determined before:

(α, β)	# of roots α with given inner product	eigenvalue
2	1	1
1	56	-1
0	126	1
-1	56	-1
-2	1	1

Thus, the trace is $1 - 56 + 126 - 56 + 1 + 8 = 24$ (the 8 is from the Cartan subalgebra). So the signature of the corresponding form on the Lie algebra is -24. We've found a third Lie algebra.

If we also look at the case when $\beta^2 = 4$, what happens? How many α with $\alpha^2 = 2$ and with given (α, β) are there? In this case, we have:

(α, β)	# of roots α with given inner product	eigenvalue
2	14	1
1	64	-1
0	84	1
-1	64	-1
-2	14	1

The trace will be $14 - 64 + 84 - 64 + 14 + 8 = -8$. This is just the split form again.

Summary: We've found 3 forms of E_8, corresponding to 3 classes in $L/2L$, with signatures 8, -24, -248, corresponding to involutions $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^{-\alpha}$ of the *compact* form. If L is the *compact* form of a simple Lie algebra, then Cartan showed that the other forms correspond exactly to the conjugacy classes of involutions in the automorphism group of L (this doesn't work if we don't start with the compact form — so always start with the compact form).

In fact, these three are the *only* forms of E_8, but we won't prove that.

Example 395 Let's go back to various forms of E_8 and figure out (guess) the fundamental groups. We need to know the maximal compact subgroups.

1. One of them is easy: the compact form is its own maximal compact subgroup. What is the fundamental group? Remember or quote the fact that for compact simple groups, $\pi_1 \cong \frac{\text{weight lattice}}{\text{root lattice}}$, which is 1. So this form is simply connected.

2. $\beta^2 = 2$ case (signature -24). Recall that there were 1, 56, 126, 56, and 1 roots α with $(\alpha, \beta) = 2, 1, 0, -1$, and -2 respectively, and there are another 8 dimensions for the Cartan subalgebra. On the $1, 126, 1, 8$ parts, the form is negative definite. The sum of these root spaces gives a Lie algebra of type E_7A_1 with a negative definite bilinear form (the 126 gives you the roots of an E_7, and the 1s are the roots of an A_1). So it a reasonable guess that the maximal compact subgroup has something to do with E_7A_1. E_7 and A_1 are not simply connected: the compact form of E_7 has $\pi_1 = \mathbb{Z}/2$ and the compact form of A_1 also has $\pi_1 = \mathbb{Z}/2$. So the universal cover of E_7A_1 has center $(\mathbb{Z}/2)^2$. Which part of this acts trivially on E_8? We look at the E_8 Lie algebra as a representation of $E_7 \times A_1$. You can read off how

it splits form the picture above: $E_8 \cong E_7 \oplus A_1 \oplus 56 \otimes 2$, where 56 and 2 are irreducible, and the centers of E_7 and A_1 both act as -1 on them. So the maximal compact subgroup of this form of E_8 is the simply connected compact form of $E_7 \times A_1/(-1,-1)$. This means that $\pi_1(E_8)$ is the same as π_1 of the compact subgroup, which is $(\mathbb{Z}/2)^2/(-1,-1) \cong \mathbb{Z}/2$. So this simple group has a nontrivial double cover (which is non-algebraic).

3. For the other (split) form of E_8 with signature 8, the maximal compact subgroup is $\mathrm{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2)$, and $\pi_1(E_8)$ is $\mathbb{Z}/2$.

You can compute any other homotopy invariants with this method.

Let's look at the 56 dimensional representation of E_7 in more detail. We had the picture

(α,β)	# of α's
2	1
1	56
0	126
-1	56
-2	1

The Lie algebra E_7 fixes these 5 spaces of E_8 of dimensions $1, 56, 126+8, 56, 1$. From this we can get some representations of E_7. The $126+8$ splits as $1+(126+7)$. But we also get a 56 dimensional representation of E_7. Let's show that this is actually an irreducible representation. Recall that in calculating $W(E_8)$, we showed that $W(E_7)$ acts transitively on this set of 56 roots of E_8, which can be considered as weights of E_7.

An irreducible representation is called *minuscule* if the Weyl group acts transitively on the weights. This kind of representation is particularly easy to work with. It is really easy to work out the character for example: just translate the 1 at the highest weight around, so every weight has multiplicity 1.

So the 56 dimensional representation of E_7 must actually be the irreducible representation with whatever highest weight corresponds to one of the vectors.

37.2 Every possible simple Lie group

We will construct them as follows: Take an involution σ of the compact form $L = L^+ + L^-$ of the Lie algebra, and form $L^+ + iL^-$. The way we constructed these was to first construct A_n, D_n, E_6, and E_7 as for E_8. Then construct the involution $\omega : e^\alpha \mapsto -e^{-\alpha}$. We get B_n, C_n, F_4, and G_2 as fixed points of the involution ω.

Kac classified all automorphisms of finite order of any compact simple Lie group. The method we'll use to classify involutions is extracted from his method. We can construct lots of involutions as follows:

1. Take any Dynkin diagram, say E_8, and select some of its verticals, corresponding to simple roots. Get an involution by taking $e^\alpha \mapsto \pm e^\alpha$ where the sign depends on whether α is one of the simple roots we've selected. However, this is not a great method. For one thing, we get a lot of repeats (recall that there are only 3, and we've found 2^8 this way).

2. Take any diagram automorphism of order 2, such as

   This gives you more involutions.

Next time, we'll see how to cut down this set of involutions. Split form of Lie algebra (we did this for A_n, D_n, E_6, E_7, E_8): $A = \bigoplus \hat{e}^\alpha \oplus L$. Compact form $A^+ + iA^-$, where $A^\pm$ eigenspaces of $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}\hat{e}^{-\alpha}$.

We talked about other involutions of the compact form. You get all the other forms this way.

The idea now is to find ALL real simple Lie algebras by listing all involutions of the compact form. We will construct all of them, but we won't prove that we have all of them.

We'll use Kac's method for classifying all automorphisms of order N of a compact Lie algebra (and we'll only use the case $N = 2$). First let's look at inner automorphisms. Write down the AFFINE Dynkin diagram

Choose n_i with $\sum n_i m_i = N$ where the m_i are the numbers on the diagram. We have an automorphism $e^{\alpha_j} \mapsto e^{2\pi i n_j/N} e^{\alpha_j}$ induces an automorphism of order dividing N. This is obvious. The point of Kac's theorem is that all inner automorphisms of order dividing N are obtained this way and are conjugate if and only if they are conjugate by an automorphism of the Dynkin diagram. We won't actually prove Kac's theorem because we just want to get a bunch of examples.

Example 396 Real forms of E_8. We've already found three, and it took us a long time. We can now do it fast. We need to solve $\sum n_i m_i = 2$ where $n_i \geq 0$; there are only a few possibilities:

The points NOT crossed off form the Dynkin diagram of the maximal compact subgroup. Thus, by just looking at the diagram, we can see what all the real forms are!

Example 397 Let's do E_7. Write down the affine diagram:

We get the possibilities

(*) The number of ways is counted up to automorphisms of the diagram.

(**) In the split real form, the maximal compact subgroup has dimension equal to half the number of roots. The roots of A_7 look like $\varepsilon_i - \varepsilon_j$ for $i, j \leq 8$ and $i \neq j$, so the dimension is $8 \cdot 7 + 7 = 56 = \frac{112}{2}$.

(***) The maximal compact subgroup is $E_6 \oplus \mathbb{R}$ because the fixed subalgebra contains the whole Cartan subalgebra, and the E_6 only accounts for 6 of the 7 dimensions. You can use this to construct some interesting representations of E_6 (the minuscule ones). How does the algebra E_7 decompose as a representation of the algebra $E_6 \oplus \mathbb{R}$?

We can decompose it according to the eigenvalues of $\mathbb{R}$. The $E_6 \oplus \mathbb{R}$ is the zero eigenvalue of $\mathbb{R}$ [why?], and the rest is 54 dimensional. The easy way to see the decomposition is to look at the roots. Remember when we computed the Weyl group we looked for vectors like

The 27 possibilities (for each) form the weights of a 27 dimensional representation of E_6. The orthogonal complement of the two nodes is an E_6 root system whose Weyl group acts transitively on these 27 vectors (we showed that these form a single orbit, remember?). Vectors of the E_7 root system are the vectors of the E_6 root system plus these 27 vectors plus the other 27 vectors. This splits up the E_7 explicitly. The two 27s form single orbits, so they are irreducible. Thus, $E_7 \cong E_6 \oplus \mathbb{R} \oplus 27 \oplus 27$, and the 27s are minuscule.

Let K be a maximal compact subgroup, with Lie algebra $\mathbb{R} + E_6$. The factor of $\mathbb{R}$ means that K has an S^1 in its center. Now look at the space G/K, where G is the Lie group of type E_7, and K is the maximal compact subgroup. It is a *Hermitian symmetric space*. Symmetric space means that it is a (simply connected) Riemannian manifold M such that for each point $p \in M$, there is an automorphism fixing p and acting as -1 on the tangent space. This looks weird, but it turns out that all kinds of nice objects you know about are symmetric spaces. Typical examples you may have seen: spheres S^n, hyperbolic space $\mathbb{H}^n$, and Euclidean space $\mathbb{R}^n$. Roughly speaking, symmetric spaces have nice properties of these spaces. Cartan classified all symmetric spaces: they are non-compact simple Lie groups modulo the maximal compact subgroup (more or less ... depending on simply connectedness hypotheses 'n such). Historically, Cartan classified simple Lie groups, and then later classified symmetric spaces, and was surprised to find the same result. Hermitian symmetric spaces are just symmetric spaces with a complex structure. A standard example of this is the upper half plane $\{x + iy | y > 0\}$. It is acted on by $SL_2(\mathbb{R})$, which acts by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau = \frac{a\tau+b}{c\tau+d}$.

Let's go back to this G/K and try to explain why we get a Hermitian symmetric space from it. We'll be rather sketchy here. First of all, to make it a symmetric space, we have to find a nice invariant Riemannian metric on it. It is sufficient to find a positive definite bilinear form on the tangent space at p which is invariant under K ... then we can translate it around. We can do this as K is compact (so you have the averaging trick). Why is it Hermitian? We'll show that there is an almost complex structure. We have S^1 acting on the tangent space of each point because we have an S^1 in the center of the stabilizer of any given point. Identify this S^1 with complex numbers of absolute value 1. This gives an invariant almost complex structure on G/K. That is, each tangent space is a complex vector space. Almost complex structures don't always come from complex structures, but this one does (it is integrable). Notice that it is a little unexpected that G/K has a complex structure (G and K are odd dimensional in the case of $G = E_7$, $K = E_6 \oplus \mathbb{R}$, so they have no hope of having a complex structure).

Example 398 Let's look at E_6, with affine Dynkin diagram

We get the possibilities

In the last one, the maximal compact subalgebra is $D_5 \oplus \mathbb{R}$. Just as before, we get a Hermitian symmetric space. Let's compute its dimension (over $\mathbb{C}$). The dimension will be the dimension of E_6 minus the dimension of $D_5 \oplus \mathbb{R}$, all divided by 2 (because we want complex dimension), which is $(78 - 46)/2 = 16$.

So we have found two non-compact simply connected Hermitian symmetric spaces of dimensions 16 and 27. These are the only "exceptional" cases; all the others fall into infinite families!

There are also some OUTER automorphisms of E_6 coming from the diagram automorphism

The fixed point subalgebra has Dynkin diagram obtained by folding the E_6 on itself. This is the F_4 Dynkin diagram. The fixed points of E_6 under the diagram automorphism is an F_4 Lie algebra. So we get a real form of E_6 with maximal compact subgroup F_4. This is probably the easiest way to construct F_4, by the way. Moreover, we can decompose E_6 as a representation of F_4. $\dim E_6 = 78$ and $\dim F_4 = 52$, so $E_6 = F_4 \oplus 26$, where 26 turns out to be irreducible (the smallest non-trivial representation of F_4 ... the only one anybody actually works with). The roots of F_4 look like $(\ldots, \pm 1, \pm 1 \ldots)$ (24 of these) and $(\pm\frac{1}{2} \cdots \pm \frac{1}{2})$ (16 of these), and $(\ldots, \pm 1 \ldots)$ (8 of them) ... the last two types are in the same orbit of the Weyl group.

The 26 dimensional representation has the following character: it has all norm 1 roots with multiplicity 1 and 0 with multiplicity 2 (note that this is not minuscule).

There is one other real form of E_6. To get at it, we have to talk about Kac's description of non-inner automorphisms of order N. The non-inner automorphisms all turn out to be related to diagram automorphisms. Choose a diagram automorphism of order r, which divides N. Let's take the standard thing on E_6. Fold the diagram (take the fixed points), and form a TWISTED affine Dynkin diagram (note that the arrow goes the wrong way from the affine F_4)

There are also numbers on the twisted diagram, but never mind them. Find n_i so that $r \sum n_i m_i = N$. This is Kac's general rule. We'll only use the case $N = 2$.

If $r > 1$, the only possibility is $r = 2$ and one n_1 is 1 and the corresponding m_i is 1. So we just have to find points of weight 1 in the twisted affine Dynkin diagram. There are just two ways of doing this in the case of E_6

one of these gives us F_4, and the other has maximal compact subalgebra C_4, which is the split form since $\dim C_4 = \#\text{roots of } F_4/2 = 24$.

Example 399 F_4. The affine Dynkin is

We can cross out one node of weight 1, giving the compact form (split form), or a node of weight 2 (in two ways), giving maximal compacts A_1C_3 or B_4. This gives us three real forms.

Example 400 G_2. We can actually draw this root system ... UCB won't supply me with a four dimensional board. The construction is to take the D_4 algebra and look at the fixed points of:

We want to find the fixed point subalgebra.

Fixed points on Cartan subalgebra: ρ fixes a two dimensional space, and has 1 dimensional eigenspaces corresponding to ω and $\bar{\omega}$, where $\omega^3 = 1$. The 2 dimensional space will be the Cartan subalgebra of G_2.

Positive roots of D_4 as linear combinations of simple roots (not fundamental weights):

There are six orbits under ρ, grouped above. It obviously acts on the negative roots in exactly the same way. What we have is a root system with six roots of norm 2 and six roots of norm 2/3. Thus, the root system is G_2:

One of the only root systems to appear on a country's national flag. Now let's work out the real forms. Look at the affine:

we can delete the node of weight 1, giving the compact form:

. We can delete the node of weight 2, giving A_1A_1 as the compact subalgebra:

... this must be the split form because there is nothing else the split form can be.

Let's say some more about the split form. What does the Lie algebra of G_2 look like as a representation of the maximal compact subalgebra $A_1 \times A_1$? In this case, it is small enough that we can just draw a picture:

We have two orthogonal A_1s, and we have leftover the stuff on the right. This thing on the right is a tensor product of the 4 dimensional irreducible representation of the horizontal and the 2 dimensional of the vertical. Thus, $G_2 = 3\times 1+1\otimes 3+4\otimes 2$ as irreducible representations of $A_1^{(\text{horizontal})}\otimes A_1^{(\text{vertical})}$.

Let's use this to determine exactly what the maximal compact subgroup is. It is a quotient of the simply connected compact group $SU(2)\times SU(2)$, with Lie algebra $A_1\times A_1$. Just as for E_8, we need to identify which elements of the center act trivially on G_2. The center is $\mathbb{Z}/2\times\mathbb{Z}/2$. Since we've decomposed G_2, we can compute this easily. A non-trivial element of the center of $SU(2)$ acts as 1 (on odd dimensional representations) or -1 (on even dimensional representations). So the element $z\times z\in SU(2)\times SU(2)$ acts trivially on $3\otimes 1+1\otimes 3+4\times 2$. Thus the maximal compact subgroup of the non-compact simple G_2 is $SU(2)\times SU(2)/(z\times z)\cong SO_4(\mathbb{R})$, where z is the non-trivial element of $\mathbb{Z}/2$.

So we have constructed $3+4+5+3+2$ (from E_8, E_7, E_6, F_4, G_2) real forms of exceptional simple Lie groups.

There are another 5 exceptional real Lie groups: Take COMPLEX groups $E_8(\mathbb{C})$, $E_7(\mathbb{C})$, $E_6(\mathbb{C})$, $F_4(\mathbb{C})$, and $G_2(\mathbb{C})$, and consider them as REAL. These give simple real Lie groups of dimensions 248×2, 133×2, 78×2, 52×2, and 14×2.