Let's now look at lifts of automorphisms of L to $\hat{L}$.

Exercise 390 Any automorphism of L preserving $(\ ,\ )$ lifts to an automorphism of $\hat{L}$

There are two special cases:

1. -1 is an automorphism of L, and we want to lift it to $\hat{L}$ explicitly. First attempt: try sending $\hat{e}^\alpha$ to $\hat{e}^{-\alpha} := (\hat{e}^\alpha)^{-1}$, which doesn't work because $a \mapsto a^{-1}$ is not an automorphism on non-abelian groups.

   Better: $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$ is an automorphism of $\hat{L}$. To see this, check

$$\begin{aligned}\omega(\hat{e}^\alpha)\omega(\hat{e}^\beta) &= (-1)^{(\alpha^2+\beta^2)/2}(\hat{e}^\alpha)^{-1}(\hat{e}^\beta)^{-1}\\ \omega(\hat{e}^\alpha\hat{e}^\beta) &= (-1)^{(\alpha+\beta)^2/2}(\hat{e}^\beta)^{-1}(\hat{e}^\alpha)^{-1}\end{aligned}$$

   which work out just right

2. If $r^2 = 2$, then $\alpha \mapsto \alpha - (\alpha, r)r$ is an automorphism of L (reflection through $r^\perp$). You can lift this by $\hat{e}^\alpha \mapsto \hat{e}^\alpha(\hat{e}^r)^{-(\alpha,r)} \times (-1)^{\binom{(\alpha,r)}{2}}$. This is a homomorphism (check it!) of order (usually) 4!

   Remark 391 Although automorphisms of L lift to automorphisms of $\hat{L}$, the lift might have larger order.

This construction works for the root lattices of A_n, D_n, E_6, E_7, and E_8; these are the lattices which are even, positive definite, and generated by vectors of norm 2 (in fact, all such lattices are sums of the given ones). What about B_n, C_n, F_4 and G_2? The reason the construction doesn't work for these cases is because there are roots of different lengths. These all occur as fixed points of diagram automorphisms of A_n, D_n and E_6. In fact, we have a *functor* from Dynkin diagrams to Lie algebras, so and automorphism of the diagram gives an automorphism of the algebra

A_{2n} doesn't really give you a new algebra: it corresponds to some superalgebra stuff.

36.1 Construction of the Lie group of a Lie algebra

It is the group of automorphisms of the Lie algebra generated by the elements $\exp(\lambda Ad(\hat{e}^\alpha))$, where λ is some real number, $\hat{e}^\alpha$ is one of the basis elements of the Lie algebra corresponding to the root α, and $Ad(\hat{e}^\alpha)(a) = [\hat{e}^\alpha, a]$. In other words,

$$\exp(\lambda Ad(\hat{e}^\alpha))(a) = 1 + \lambda[\hat{e}^\alpha, a] + \frac{\lambda^2}{2}[\hat{e}^\alpha, [\hat{e}^\alpha, a]].$$

and all the higher terms are zero. To see that $Ad(\hat{e}^\alpha)^3 = 0$, note that if β is a root, then $\beta + 3\alpha$ is not a root (or 0).

Warning 392 In general, the group generated by these automorphisms is not the whole automorphism group of the Lie algebra. There can be extra diagram (or graph) automorphisms, field automorphisms induced (obviously) by automorphism of the field, and "diagonal" automorphisms, such as the automorphisms of $SL_n(\mathbb{Q})$ induced by conjugation by diagonal matrices of $GL_n(\mathbb{Q})$. Moreover there are some strange extra diagram automorphisms of B_2 and F_4 in characteristic 2, and of G_2 in characteristic 3. (Informally, one can ignore the arrow on a bond of a Dynkin diagram if the characteristic of the field is the number of bonds.)

We get some other things from this construction. We can get simple groups over finite fields: note that the construction of a Lie algebra above works over any commutative ring (e.g. over $\mathbb{Z}$). The only place we used division is in $\exp(\lambda Ad(\hat{e}^\alpha))$ (where we divided by 2). The only time this term is non-zero is when we apply $\exp(\lambda Ad(\hat{e}^\alpha))$ to $\hat{e}^{-\alpha}$, in which case we find that $[\hat{e}^\alpha, [\hat{e}^\alpha, \hat{e}^{-\alpha}]] = [\hat{e}^\alpha, \alpha] = -(\alpha, \alpha)\hat{e}^\alpha$, and the fact that $(\alpha, \alpha) = 2$ cancels the division by 2. So we can in fact construct the E_8 group over *any* commutative ring. With more effort we in fact get group schemes over $\mathbb{Z}$.See Steinberg's notes or the book by Carter on Finite Simple Groups for more details.