47 Construction of Lie algebras from a root lattice

The root space decomposition of a Lie algebra suggests the following construction of a Lie algebra from its root system. Take the direct sum of the (dual of the) root lattice with a 1-dimensional vector space generated by a special element for each root. However when we try to write down the Lie bracket for this algebra we run into the following sign problem: suppose that α, β, and $\alpha+\beta$ are roots, with corresponding elements e_α, e_β, $e_{\alpha+\beta}$. It seems natural to define $[e_\alpha, e_\beta] = e_{\alpha+\beta}$ The problem is that the left hand side changes sign when α and β are switched, while the right hand side does not. In fact there is in general no functorial way to define a Lie algebra from its root lattice. One way to see this is that if we had such a functor, then the automorphism group, and the Weyl group, of the root lattice would act on the Lie algebra. However in general the Weyl group does not have a nice action on the Lie algebra: it is a subquotient of the Lie group, not a subgroup. This can be seen even for $SL_n(R)$, where order 2 reflections of the Weyl group lift to order 4 elements of the Lie group. The best we can do is find a subgroup of the form $2^n.W$ inside the Lie group.

We can see this going wrong even in the case of $\mathfrak{sl}_2(\mathbb{R})$. Remember that the Weyl group is $N(T)/T$ where $T = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ and $N(T) = T \cup \left(\begin{smallmatrix} 0 & b \\ -b^{-1} & 0\end{smallmatrix}\right)$, and this second part is stuff having order 4, so we cannot possibly write this as a semi-direct product of T and the Weyl group.

So the Weyl group is not usually a subgroup of $N(T)$. The best we can do is to find a group of the form $2^n \cdot W \subseteq N(T)$ where n is the rank. For example, let's do it for $SL(n+1,\mathbb{R})$ Then $T = diag(a_1, \ldots, a_n)$ with $a_1 \cdots a_n = 1$. Then we take the normalizer of the torus to be $N(T)$ =all permutation matrices with ± 1's with determinant 1, so this is $2^n \cdot S_n$, and it does not split. The problem we had with signs can be traced back to the fact that this group doesn't split.

We can construct the Lie algebra from something acted on by $2^n \cdot W$ (but not from something acted on by W). We take a central extension of the lattice by a group of order 2. Notation is a pain because the lattice is written additively and the extension is nonabelian, so we want it to be written multiplicatively. Write elements of the lattice in the form e^α formally, so we have converted the lattice operation to multiplication. We will use the central extension

$$1 \to \pm 1 \to \hat{e}^L \to \underbrace{e^L}_{\cong L} \to 1$$

We want $\hat{e}^L$ to have the property that $\hat{e}^\alpha \hat{e}^\beta = (-1)^{(\alpha,\beta)} \hat{e}^\beta \hat{e}^\alpha$, where $\hat{e}^\alpha$ is something mapping to e^α. What do the automorphisms of $\hat{e}^L$ look like? We get

$$1 \to \underbrace{(L/2L)}_{(\mathbb{Z}/2)^{\mathrm{rank}(L)}} \to \mathrm{Aut}(\hat{e}^L) \to \mathrm{Aut}(e^L)$$

for $\alpha \in L/2L$, we get the map $\hat{e}^\beta \to (-1)^{(\alpha,\beta)} \hat{e}^\beta$. The map turns out to be onto, and the group $\mathrm{Aut}(e^L)$ contains the reflection group of the lattice. This extension is usually non-split.

Now the Lie algebra is $L \oplus \{1$ dimensional spaces spanned by $(\hat{e}^\alpha, -\hat{e}^\alpha)\}$ for $\alpha^2 = 2$ with the convention that $-\hat{e}^\alpha$ (-1 in the vector space) is $-\hat{e}^\alpha$ (-1 in the group $\hat{e}^L$). Now define a Lie bracket by the "obvious rules" $[\alpha, \beta] = 0$ for $\alpha, \beta \in L$ (the Cartan subalgebra is abelian), $[\alpha, \hat{e}^\beta] = (\alpha, \beta)\hat{e}^\beta$ ($\hat{e}^\beta$ is in the root space of β), and $[\hat{e}^\alpha, \hat{e}^\beta] = 0$ if $(\alpha, \beta) \geq 0$ (since $(\alpha+\beta)^2 > 2$), $[\hat{e}^\alpha, \hat{e}^\beta] = \hat{e}^\alpha\hat{e}^\beta$ if $(\alpha, \beta) < 0$ (product in the group $\hat{e}^L$), and $[\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}] = \alpha$.

Theorem 441 *Assume L is positive definite. Then this Lie bracket forms a Lie algebra (so it is skew and satisfies Jacobi).*

Proof The proof is easy but tiresome, because there are a lot of cases.

We check the Jacobi identity: We want $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$

1. all of a, b, c in L. Trivial because all brackets are zero.
2. two of a, b, c in L. Say α, β, e^γ

$$\underbrace{[[\alpha, \beta], e^\gamma]}_{0} + \underbrace{[[\beta, e^\gamma], \alpha]}_{(\beta,\alpha)(-\alpha,\beta)e^\gamma} + [[e^\gamma, \alpha], \beta]$$

   and similar for the third term, giving a sum of 0.

3. one of a, b, c in L. $\alpha, e^\beta, e^\gamma$. e^β has weight β and e^γ has weight γ and $e^\beta e^\gamma$ has weight $\beta + \gamma$. So check the cases, and we get Jacobi:

$$\begin{aligned}
[[\alpha, e^\beta], e^\gamma] &= (\alpha, \beta)[e^\beta, e^\gamma] \\
[[e^\beta, e^\gamma], \alpha] &= -[\alpha, [e^\beta, e^\gamma]] = -(\alpha, \beta+\gamma)[e^\beta, e^\gamma] \\
[[e^\gamma, \alpha], e^\beta] &= -[[\alpha, e^\gamma], e^\beta] = (\alpha, \gamma)[e^\beta, e^\gamma],
\end{aligned}$$

   so the sum is zero.

4. none of a, b, c in L. This is the most tedious case, $e^\alpha, e^\beta, e^\gamma$. We can reduce it to two or three cases. We make our cases depending on (α, β), (α, γ), (β, γ).
   (a) if 2 of these are 0, then all the $[[*, *], *]$ are zero.
   (b) $\alpha = -\beta$. By case a, γ cannot be orthogonal to them, so say $(\alpha, \gamma) = 1$ $(\gamma, \beta) = -1$; adjust so that $e^\alpha e^\beta = 1$, then calculate

$$\begin{aligned}
[[e^\gamma, e^\beta], e^\alpha] - [[e^\alpha, e^\beta], e^\gamma] + [[e^\alpha, e^\gamma], e^\beta] &= e^\alpha e^\beta e^\gamma - (\alpha, \gamma)e^\gamma + 0 \\
&= e^\gamma - e^\gamma = 0.
\end{aligned}$$

   (c) $\alpha = -\beta = \gamma$, which is easy because $[e^\alpha, e^\gamma] = 0$ and $[[e^\alpha, e^\beta], e^\gamma] = -[[e^\gamma, e^\beta], e^\alpha]$
   (d) We have that each of the inner products is 1, 0 or -1. If some $(\alpha, \beta) = 1$, all brackets are 0.

□

We had two cases left:

$$[[e^\alpha, e^\beta], e^\gamma] + [[e^\beta, e^\gamma], e^\alpha] + [[e^\gamma, e^\alpha], e^\beta] = 0$$

– $(\alpha, \beta) = (\beta, \gamma) = (\gamma, \alpha) = -1$, in which case $\alpha + \beta + \gamma = 0$. then $[[e^\alpha, e^\beta], e^\gamma] = [e^\alpha e^\beta, e^\gamma] = \alpha + \beta$. By symmetry, the other two terms are $\beta + \gamma$ and $\gamma + \alpha$;the sum of all three terms is $2(\alpha + \beta + \gamma) = 0$.

– $(\alpha, \beta) = (\beta, \gamma) = -1$, $(\alpha, \gamma) = 0$, in which case $[e^\alpha, e^\gamma] = 0$. We check that $[[e^\alpha, e^\beta], e^\alpha] = [e^\alpha e^\beta, e^\gamma] = e^\alpha e^\beta e^\gamma$ (since $(\alpha + \beta, \gamma) = -1$). Similarly, we have $[[e^\beta, e^\gamma], e^\alpha] = [e^\beta e^\gamma, e^\alpha] = e^\beta e^\gamma e^\alpha$. We notice that $e^\alpha e^\beta = -e^\beta e^\alpha$ and $e^\gamma e^\alpha = e^\alpha e^\gamma$ so $e^\alpha e^\beta e^\gamma = -e^\beta e^\gamma e^\alpha$; again, the sum of all three terms in the Jacobi identity is 0.

This concludes the verification of the Jacobi identity, so we have a Lie algebra.

Is there a proof avoiding case-by-case check? Yes, but it is actually more work. We really have functors from double covers of lattices to vertex algebras, and from vertex algebras to Lie algebras. However it takes several weeks to define vertex algebras, though it you do you get constructions for a lot more Lie algebras because this works even if the lattice is not positive definite. In fact, the construction we did was the vertex algebra approach, with all the vertex algebras removed. So there is a more general construction which gives a much larger class of infinite dimensional Lie algebras.

Now we should study the double cover $\hat{L}$, and in particular prove its existence. Given a Dynkin diagram, we can construct $\hat{L}$ as generated by the elements e^{α_i} for α_i simple roots with the given relations. It is easy to check that we get a surjective homomorphism $\hat{L} \to L$ with kernel generated by z with $z^2 = 1$. What's a little harder to show is that $z \neq 1$ (i.e., show that $\hat{L} \neq L$). The easiest way to do it is to use cohomology of groups, but since we have such an explicit case, we'll do it bare hands:
Problem: Given Z, H groups with Z abelian, construct central extensions

$$1 \to Z \to G \to H \to 1$$

(where Z lands in the center of G). Let G be the set of pairs (z, h), and set the product $(z_1, h_1)(z_2, h_2) = (z_1 z_2 c(h_1, h_2), h_1 h_2)$, where $c(h_1, h_2) \in Z$ ($c(h_1, h_2)$ will be a cocycle in group cohomology). We obviously get a homomorphism by mapping $(z, h) \mapsto h$. If $c(1, h) = c(h, 1) = 1$ (normalization), then $z \mapsto (z, 1)$ is a homomorphism mapping Z to the center of G. In particular, $(1, 1)$ is the identity. We'll leave it as an exercise to figure out what the inverses are. When is this thing *associative*? Let's just write everything out:

$$\begin{aligned}
\big((z_1, h_1)(z_2, h_2)\big)(z_3, h_3) &= (z_1 z_2 z_3 c(h_1, h_2) c(h_1 h_2, h_3), h_1 h_2 h_3) \\
(z_1, h_1)\big((z_2, h_2)(z_3, h_3)\big) &= (z_1 z_2 z_3 c(h_1, h_2 h_3) c(h_2, h_3), h_1 h_2 h_3)
\end{aligned}$$

so we must have

$$c(h_1, h_2) c(h_1 h_2, h_3) = c(h_1 h_2, h_3) c(h_2, h_3).$$

This identity is actually very easy to satisfy in one particular case: when c is bimultiplicative:

$$c(h_1, h_2 h_3) = c(h_1, h_2) c(h_1, h_3)$$

and

$$c(h_1 h_2, h_3) = c(h_1, h_3) c(h_2, h_3)$$

. That is, we have a map $H \times H \to Z$. Not all cocycles come from such maps, but this is the case we care about.

To construct the double cover, let $Z = \pm 1$ and $H = L$ (free abelian). If we write H additively, we want c to be a bilinear map $L \times L \to \pm 1$. It is really easy to construct bilinear maps on free abelian groups. Just take any basis $\alpha_1, \ldots, \alpha_n$ of L, choose $c(\alpha_1, \alpha_j)$ arbitrarily for each i, j and extend c via bilinearity to $L \times L$. In our case, we want to find a double cover $\hat{L}$ satisfying $\hat{e}^\alpha \hat{e}^\beta = (-1)^{(\alpha,\beta)} \hat{e}^\beta \hat{e}^\alpha$ where $\hat{e}^\alpha$ is a lift of e^α. This just means that $c(\alpha, \beta) = (-1)^{(\alpha,\beta)} c(\beta, \alpha)$. To satisfy this, just choose $c(\alpha_i, \alpha_j)$ on the basis $\{\alpha_i\}$ so that $c(\alpha_i, \alpha_j) = (-1)^{(\alpha_i,\alpha_j)} c(\alpha_j, \alpha_i)$. This is trivial to do as $(-1)^{(\alpha_i,\alpha_i)} = 1$. Notice that this uses the fact that the lattice is even. There is no canonical way to choose this 2-cocycle (otherwise, the central extension would split as a product), but all the different double covers are isomorphic because we can specify $\hat{L}$ by generators and relations. Thus, we have constructed $\hat{L}$ (or rather, verified that the kernel of $\hat{L} \to L$ has order 2, not 1).