We write $T_{\geq j}$ for the tableau formed by the columns $j, j+1, \ldots$ of T, and defined $T_{>j}$, $T_{<j}$, and so on in a similar way. The weight $\omega(T)$ of a tableau T is (number of 1's in T, number of 2's, ...)$\in \mathbb{Z}^n$. So the Young diagram $\lambda + \omega(T)$ is formed by adding a box to the first row for every 1 in T, a box to the second row for every 2 in T, and so on. The Weyl vector ρ is $(n-1, n-2, \cdots, 0) \in \mathbb{Z}^n$. We think of partitions into at most n parts as non-increasing sequences in $\mathbb{Z}^n$.

We define a_λ to be $\sum_{w \in S_n} \epsilon(w) x^{w(\lambda)}$, where $\epsilon = \pm 1$ is the sign of the permutation w. In particular a_λ is alternating under permutations of λ and vanishes if two elements of λ are equal.

The Littlewood-Richardson rule will follow easily from the following result:

Theorem 377

$$a_{\lambda+\rho} s_\mu = \sum_T a_{\lambda+\omega(T)+\rho}$$

where the sum is over all semistandard tableaux T of shape μ such that for all j, $\lambda + \omega(T_{\geq j})$ is a partition.

Proof Since s_μ is symmetric, we have

$$a_{\lambda+\rho} s_\mu = \sum_T a_{\lambda+\omega(T)+\rho}$$

where the sum is over all semistandard tableaux T. The core part of the proof is to show that the terms such that for some j, $\lambda + \omega(T_{\geq j})$ is not a partition cancel out in pairs or are zero, which we will do using Bender-Knuth involutions to pair them off.

Fix some j and k. We concentrate on the tableaux T such that $\lambda + \omega(T_{\geq j})$ is not a partition and $\lambda + \omega(T_{\geq i})$ is a partition for any $i > j$. We further restrict to the set X of T such that k is the smallest number with $\lambda_k + \omega_k(T_{\geq j} > \lambda_{k+1} + \omega_{k+1}(T_{\geq j}$. This implies that column j of T contains a $k+1$ and does not contain a k, and that $T_{>j}$ has the same number of ks and $(k+1)$s.

We define an involution $*$ on this set of T by fixing $T_{\geq j}$ and acting as a Bender-Knuth involution $(k, k+1)$ on $T_{<j}$. This takes elements of X to elements of X because $*$ does not change $T_{\geq j}$, and row j of T does not contain k (so that $*$ keeps the rows monotonic).

We check that the term corresponding to T cancels out with the term corresponding to $*T$. The transposition $(k, k+1)$ of S_n fixes $\lambda + \omega(T_{\geq j} + \rho$. It also maps $\omega(T_{<j}$ to $\omega_(*T_{<j})$, so maps $\lambda + \omega(T) + \rho$ to $\lambda + \omega(T) + \rho$. Also $a_{\lambda+\omega(T)+\rho} = -a_{\lambda+\omega(*T)+\rho}$ as a is alternating under permutations of S_n, so the terms corresponding to T and $*T$ cancel (it is possible that $*T = T$ in which case the term is 0). $\square$

Corollary 378

$$s_\mu = a_{\mu+\rho}/a_\rho$$

Proof This is just the special case $\lambda = 0$. $\square$

Corollary 379 *(The Littlewood-Richardson rule)*

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

where $c^{\nu}_{\lambda\mu}$ is the number semistandard tableaux T of shape μ such that $\nu = \lambda + \omega(T)$ and for all j, $\lambda + \omega(T_{\geq j})$ is a partition.

Proof This follows by combining theorem ?? with corollary ??. $\square$

Warning 380 It is hard to avoid errors when doing hand calculations with the Littlewood-Richardson rule. An easy error to make is forgetting to include some tableaux. As a check, one can do the calculation for the product in reverse order, or for the transpose of the permutations (or better still use a computer).

As the proof shows, it is rather easy to find an expression for $S_\lambda S_\mu$ as a sum of terms S_ν possibly with negative coefficients. The tricky part of the proof of the Littlewood-Richardson rule is to pair off the terms with negative coefficients with terms with positive coefficients, so that the final sum has only positive coefficients.

Example 381 We work out $S_{21}S_{21}$. There are 8 tableaux in the sum, given by $\begin{smallmatrix}1&1\\2\end{smallmatrix}$ $\begin{smallmatrix}1&1\\3\end{smallmatrix}$ $\begin{smallmatrix}1&2\\2\end{smallmatrix}$ $\begin{smallmatrix}1&2\\3\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2\end{smallmatrix}$ $\begin{smallmatrix}1&3\\4\end{smallmatrix}$ $\begin{smallmatrix}2&3\\3\end{smallmatrix}$ $\begin{smallmatrix}2&3\\4\end{smallmatrix}$. So $S_{21}S_{21} = S_{42} + S_{411} + S_{33} + 2S_{321} + S_{3111} + S_{222} + S_{2211}$.

Example 382 We use the Littlewood-Richardson rule to work out the decomposition of $V \otimes V$ into irreducible representations, where V is the 8-dimensional adjoint representation of SU_3. The partition corresponding to V is 21, which in general for GL_n corresponds to the non-trivial component of the tensor product of the n-dimensional representation and its alternating square. By the previous example, $V \otimes V$ decomposes into 8 irreducible representations if $n \geq 4$. We have to make two changes to find the decomposition for SU_3. First of all, any Young diagram with more than 3 rows corresponds to the zero representation, which eliminates 3111 and 2211. Second, since we are working with SU rather than U, any diagram with exactly 3 rows abc is equivalent to $(a-1)(b-1)(c-1)$, so 411 becomes 3, 321 becomes 21, and 222 becomes the empty Young diagram. So $V \otimes V$ decomposes into a sum of 6 irreducible representations corresponding to the partitions (42), (3), (33), (21), (21), (0).

Exercise 383 Show that the Littlewood-Richardson rule implies Pieri's formula: $S_\lambda S_n = \sum_\nu S_\nu$, where the sum is over all partitions ν obtained from λ by adding n elements with at most 1 in each column.

Exercise 384 Check the (corrected) example given by Littlewood and Richardson: $S_{431}S_{221} = S_{652} + S_{6511} + S_{643} + 2S_{6421} + S_{64111} + S_{6331} + S_{6322} + S_{63211} + S_{553} + 2S_{5521} + S_{55111} + 2S_{5431} + 2S_{5422} + 3S_{54211} + S_{541111} + S_{5332} + S_{53311} + 2S_{53221} + S_{532111} + S_{4432} + S_{44311} + 2S_{44221} + S_{442111} + S_{43321} + S_{43222} + S_{432211}$

Exercise 385 If $\mu \leq \lambda$ we define the skew Schur polynomial $S_{\lambda/\mu}$ as the sum $\sum_T x^{\omega(T)}$ over all semistandard Tableaux of shape λ/μ (the Young diagram of λ with the boxes in μ removed). Prove that this is a symmetric polynomial. Prove that

Hook formula??

35 Combinatorics of Young diagrams

RSK, Jeu de taquin, Plactic monoid, Knuth equivalence

The plectic monoid is generated by a totally ordered set, usually taken to be the positive integers, subject to the Knuth relations:

- $acb = cab$ if $a \leq b < c$
- $bac = bca$ if $a < b \leq c$

The Knuth relations say roughly that the exchange of two adjacent elements a and c is catalyzed by an element next to them that is between a and c under the total order. The relations when equality holds between two of the letters are harder to remember: the relations have the property that the triple on either side of the equality is never an increasing sequence.

For any semistandard tableau we can get an element of the plectic monoid by listing its rows starting with the lowest (in the English convention); for example the tableau $\begin{smallmatrix} 1 & 1 & 2 & 3 & 3 & 4 \\ 2 & 3 & 5 & & & \\ 3 & 7 & & & & \end{smallmatrix}$ corresponds to the word 37235112334.

The key result is the following theorem, which identifies elements of the plectic monoid with semistandard tableaux, and in particular shows that the semistandard tableaux form a monoid.

Theorem 386 *Every element of the plectic monoid is represented by a unique semistandard tableau.*

Proof We first show that every word is Knuth equivalent to the word of a semistandard tableau. For this it is sufficient to show that if we multiply a semistandard tableau on the right by a generator x we still get a semistandard tableau. This operation is called Schensted insertion and works as follows. If x is at least as large as the rightmost element of the first row we just add it to the end. Otherwise we move it to the left in the first row using a Knuth relation until it reaches yz with $y \leq x$, $z > x$ (so z is the leftmost element greater than x). Now we can move z to the left of the first row using Knuth relations. If z is at least the rightmost element of the second row we leave it there, otherwise we repeat what we did on the first row with z instead of x. Continuing in this way we obtain a semistandard tableau whose word is Knuth equivalent to the original word.

Now we have to show that any word is equivalent to at most one semistandard tableau. We start by observing that Knuth equivalence preserves the length of the longest increasing sequence. Since in a semistandard tableau the length of the longest increasing sequence is the first row, this shows that the length of the first row of the tableau is determined. More generally, Knuth equivalence preserves the maximum of the sum of the lengths of k disjoint increasing sequences for any k. For a semistandard tableau, this is the sum of the lengths of the first k rows (as there cannot be more than k elements from any column) so the shape of the tableau is determined bu the word.

To complete the proof we have to show that position of each letter of the word in the tableau is determined. Consider the rightmost largest element x in the word. It can never catalyze the exchange of two letters, so the word with x removed is Knuth equivalent to the tableau with x removed. By induction on length the partition of the tableau with x removed is uniquely determined, and is the partition of the word with one box removed, which must therefore be the place where x has to be inserted. □

Exercise 387 Show that there is a polynomial-time algorithm to find the maximal total length of k disjoint increasing subsequences of a given finite sequence.

Exercise 388 Prove the ErdsSzekeres theorem that a sequence of length $mn+1$ contains an increasing sequence of length $m+1$ or a decreasing sequence of length $n+1$. (Let a_i and b_i be the lengths of maximal increasing and decreasing sequences ending at position i. Show that the $mn+1$ pairs (a_i, b_i) are all distinct.)

The distribution of the length of the longest increasing sequence of a long random sequence approaches the TracyWidom distribution, which can be expressed in terms of Painleve functions.