33 Schur-Weyl duality

In this section V will be a complex vector space, and we will be studying complex representations of the symmetric group and GL_V.

The simplest case of Schur-Weyl duality is the decomposition of $V \otimes V$ into the sum of the symmetric square S^2V and the alternating square $\Lambda^2 V$. In terms of representation theory this can be interpreted as follows. The space $V \otimes V$ is a representation of $GL_V \times S_2$ where GL_V acts on $V \otimes V$ by acting on each factor, and the symmetric group S_2 acts by permuting the two factors of V. Then V splits up as the sum of two irreducible representations $S^2V \oplus \Lambda^2 V$ of $GL_V \times S_2$. This gives a correspondence between representations of S_2 and GL_V, with the trivial and alternating representations of S_2 corresponding to the symmetric square and alternating square representations of GL_V.

The Schur-Weyl correspondence extends this from S_2 to the symmetric group S_n on n points. This time we use the representation of $GL_V \times S_n$ on the tensor product $V \otimes V \otimes \cdots \otimes V$, where GL_V again acts in the standard way on each factor, and the symmetric group permutes the factors. The key point is that each of GL_V and S_n generate each others commutators. This implies that $V \otimes V \otimes \cdots \otimes V$ is a direct sup of representations $\oplus A_i \otimes B_i$ where A_i is an irreducible representation of GL_V, B_i is an irreducible representation of S_n, and all the A_i are distinct and all the B_i are distinct. So we get a correspondence between some representations A_i of GL_V and some representations B_i of S_n.

For any particular choice of n and V we do not usually get a 1:1 correspondence between representations of GL_V and S_n: one problem is that GL_V has an infinite number of irreducible representations if V is non-trivial, while S_n has only a finite number. Another problem is that we can take inverse powers of the determinant of GL_n. However we almost get a 1:1 correspondence if we stabilize: if we let the dimension of V become large enough we get all representations of S_n, and if we let n become large we get all representations of GL_V up to powers of the determinant.

Example 368 Suppose that V has dimension 2. Then Schur-Weyl duality for the tensor product of n copies of V gives a correspondence between representations of $SL_2(\mathbb{C})$ and S_n, where the trivial representation of S_n corresponds to the $n+1$-dimensional irreducible representation of $SL_2(\mathbb{C})$. So we pick up all irreducible representations of $SL_2(\mathbb{C})$ by just using the trivial representation of the symmetric group and letting n tend to infinity.

Exercise 369 If W is a complex vector space, show that the space of elements of $W \otimes W \otimes \cdots \otimes W$ fixed by the symmetric group S_n is spanned by elements of the form $w \otimes w \otimes \cdots \otimes w$.

The key result needed for Schur-Weyl duality is the following:

Theorem 370 *Any endomorphism of $V \otimes V \otimes \cdots \otimes V$ that commutes with S_n is a linear combination of endomorphisms given by elements of $GL_n(V)$.*

Proof We want to show that the space of elements of $\mathrm{End}(V^{\otimes n})$ fixed by the symmetric group is spanned by GL_V. But $\mathrm{End}(V^{\otimes n}) = \mathrm{End}(V)^{\otimes n}$ so by the previous exercise the space of elements fixed by S_n is spanned by elements of the form $w \otimes w \otimes \cdots \otimes w$ for $w \in \mathrm{End}(V)$, or in other words any endomorphism of

$V^{\otimes n}$ commuting with S_n is a linear combination of elements of $\mathrm{End}(V)$ acting on it. Since GL_V is dense in $\mathrm{End}(V)$ this proves the theorem. $\square$

Theorem 371 *Suppose that M is an algebra of endomorphisms of $H = \mathbb{C}^n$ containing 1 and closed under taking adjoints. Then $M = M''$*

Proof Suppose $v \in H$. Then Mv is fixed by M'. Also we can write H as a direct sum $H = Mv \oplus Mv^{\perp}$, and the orthogonal projection e onto Mv is in M' as M is closed under adjoints. So M'' maps Mv to Mv as it commutes with e. So $M''v = Mv$. (It is obvious that $M''v \subseteq Mv$ because $1 \in M$.)

Now look at the action of M on $H \oplus H \cdots \oplus H$ (n copies of H). The things commuting with M are just n by n matrices with coefficients in M', so the double commutant M'' is the same as for M acting on H. So $M(v_1 \oplus \cdots \oplus v_n) = M''(v_1 \oplus \cdots \oplus v_n)$. In other words, for any finite number of vectors, and any element of M, we can find an element of M'' having the same effect on these vectors. Since H is finite-dimensional this proves that $M'' = M$.

$\square$

Exercise 372 Find a subalgebra A of $M_2(\mathbb{C})$ containing 1 such that A'' is not equal to A.

Exercise 373 Suppose that M is a von Neumann algebra on a Hilbert space H (possibly infinite dimensional). This means that M is an algebra of bounded operators containing 1 that is weakly closed and closed under taking adjoints. Show that $M'' = M$.

Exercise 374 Suppose that M is any collection of functions from a set H to itself. Show that $M \subseteq M''$ and $M' = M'''$.

Two algebras acting on a vector space each of which is the centralizer of the other occurs quite often, and can be a very powerful technique for constructing representations of groups. For example, a von Neumann algebra M can be defined as a *-algebra of endomorphisms of a Hilbert space such that $M = M''$, where $'$ means take the commutant. The theory of dual reductive pairs depends on finding two subgroups of a metaplectic group each of which generates the commutant of the other acting on the metaplectic representation. Some of the work on the Langlands program that tries to associate a representation of a reductive group to a representation of a Galois group tries to do this by finding a representation of (reductive group) times (Galois group) such that each generates the commutator of the other, in which case one can hope to get a suitable correspondence between their representations.

34 Littlewood-Richardson rule

Given two representations of $U(n)$ we would like to decompose their tensor product into a sum of irreducible representations. This is solved by the Littlewood-Richardson rule. Irreducible polynomial representations correspond to partitions λ with at most n rows, and the tensor product of partitions corresponds to taking products of the Schur functions (which are essentially the characters of the representations). So we want to write the product $S_\lambda S_\mu$ of any two Schur polynomials explicitly as a linear combination $\sum_\nu c^\nu_{\lambda\mu} S_\nu$ of Schur polynomials. The numbers $c^\nu_{\lambda\mu}$ are called Littlewood-Richardson coefficients, and the Littlewood-Richardson rule is a combinatorial rule for calculating them.

The following proof of the Littlewood-Richardson rule (from Stembridge) is short but rather mysterious. It is really a special case of a more general result proved using crystal graphs. It may give a misleading idea of how hard the Littlewood Richardson rule was to prove: it took four decades to find the first complete proof, and several further decades for find some of the more recently short proofs. Altogether more than a dozen mathematicians contributed significantly to finding the proof given here.

Recall that a semistandard tableau is an assignment of positive integers to a Young diagram such that all rows are non-strictly increasing and all columns are strictly increasing.

The Bender-Knuth involution is an involution depending on a pair of consecutive integers $(k, k+1)$ that acts on the semistandard tableaux, and exchanges the numbers of ks and $(k+1)$s. It acts as follows. First pair off as many ks as possible with a $k+1$ below them. The leftover k s and $(k+1)$s form several disjoint rows of the form $kkk\cdots k(k+1)\cdots(k+1)$. If such a row has a copies of k and b copies of $k+1$, change some entries so it has b copies of k and a copies of $k+1$. This produces another semistandard tableau where the number of copies of k and $k+1$ has been exchanged.

Definition 375 *The Schur function s_λ is defined to be $\sum_T x^{\omega(T)}$, where the sum is over all semistandard tableaux of weight λ.*

Lemma 376 *The Schur functions are symmetric polynomials.*

Proof The Bender-Knuth involutions show that the number of tableaux of some weight is invariant under permutations of the weight. $\square$