32.2 Representations of the symmetric groups

We can now describe the characters of irreducible representations of symmetric groups in terms of the ring of symmetric functions. The idea is that we identify class functions on S_n with homogeneous functions of degree n by the Frobenius characteristic map taking a permutation of shape λ in S_n to the symmetric function $p_\lambda/n!$. This identifies the rational class functions with rational symmetric functions. Under this identification, the characters of irreducible representations correspond to Schur polynomials, and conjugacy classes of cycle shape λ correspond to p_λ/z_λ, so the character table of S_n is given by expressing the Schur polynomials as linear combinations of the symmetric functions p_λ/z_λ.

We prove this in several steps as follows:

1. Show that h_n corresponds to the trivial representation of S_n
2. Show that all homogeneous symmetric functions correspond to virtual representations, by showing that those that do are closed under products. In particular Schur polynomials correspond to generalized characters.
3. By the orthogonality relations, the characters are, up to sign, just the generalized characters of norm 1. So we show that Schur functions have norm 1, so they are irreducible characters up to sign.
4. Show that Schur polynomials are irreducible characters by showing the sign is positive.

Lemma 359 *The symmetric function h_n is the character of the trivial representation of S_n.*

Proof This is similar to the proof of Newton's identities. We have to show that $h_n = \sum_{|\lambda|=n} p_\lambda/z_\lambda$. This follows from $H(x) = \exp \int P(x)dx/x$. □

Exercise 360 Check this explicitly for $n = 3$.

Lemma 361 *If the symmetric functions a and b correspond to representations V and W of S_m and S_n, then ab corresponds to the representation*

$$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} V \otimes W$$

Proof This follows from the Frobenius formula for the character of an induced representation, which states that the character of $Ind_H^G(V)$ is obtained from the character of V by smearing it over G. □

Corollary 362 *Every homogeneous symmetric function is the character of a generalized representation of a symmetric group.*

Proof The symmetric functions for which this is true include h_n and are closed under addition and multiplication. The corollary now follows since the symmetric functions h_n generate the ring of all symmetric functions. □

Lemma 363 *The Cauchy matrix with entries* $1/(x_i - y_j)$ *has determinant*

$$\frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i - y_j)}$$

Proof If we mutiply the Cauchy determinant by $\prod_{i,j}(x_i - y_j)$ we get a polynomial of degree $n(n-1)$. It vanishes whenever two of the x_i or two of the y_i are equal, so must be divisible by the degree $n(n-1)$ polynomial $\prod_{i<j}(x_i - x_j)(y_i - y_j)$. As the degrees are the same, these two polynomials must be the same up to a constant. □

Exercise 364 Evaluate the Hilbert determinant with entries $1/(i+j-1)$ for $1 \le i, j \le n$ by expressing it in terms of a suitable Cauchy determinant.

Theorem 365 *The Schur polynomials* $s_\lambda = a_{\lambda+\rho}/a_\rho$ *form an orthonormal basis of the symmetric functions.*

Proof We have to show that

$$\prod(1 - x_i y_i)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

We will do this by evaluating the Cauchy matrix with entries $(1 - x_i y_j)^{-1}$ in two different ways. On one hand, by the previous lemma (changing x_i to $1/x_i$) it is equal to

$$\frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(1 - x_i y_j)}.$$

On the other hand, if we expand $(1 - x_i y_j)^{-1}$ as $\sum_{k \ge 0} x_i^k y_j^k$ we see that the determinant is

$$\sum \pm x_1^{\lambda_1} x_2^{\lambda_2} \cdots y_1^{\mu_1} y_2^{\mu_2} \cdots$$

where the λ_i are a permutation of the μ_i. This is equal to

$$\sum_\lambda a_\lambda(x) a_\lambda(y)$$

where a_λ is the determinant of the matrix with entries $x_i^{\lambda_j}$. Defining $s_\lambda = a_{\lambda+\rho}/a_\rho$ and using the fact that $a_\rho(x) = \prod_{i<j}(x_i - x_j)$ by the Vandermonde identity, putting everything together proves the identity stated in the theorem. □

We are now almost finished, since the Schur polynomials are generalized characters of norm 1 and are therefore irreducible characters up to sign. So we just have to pin down a the sign. (The reason why there is a sign problem can be understood as follows. The construction of the generalized characters is really a special case of a more general construction where the generalized characters appear in an alternating sum of cohomology groups. At most one of these cohomology groups is nonzero, so the sign depends on whether it is an even or an odd cohomology group.)

Lemma 366

$$\langle s_\lambda, p_1^{|\lambda|} \rangle > 0$$

Proof

□

Example 367 For the symmetric group S_3, the conjugacy classes correspond to $p_1^3/3! = e_1^3/6$, $p_1p_2/2 = e_1^3/2 - e_1e_2$, and $p_3/3 = e_1^3/3 - e_1e_2 + e_3$. The characters correspond to $s_{1^3} = e_1^3 - 2e_1e_2 + e_3$, $s_{12} = e_1e_2 - e_3$, and $s_3 = e_3$. The coefficients expressing the Schur functions in terms of the Newton functions are just the coefficients of the character table of S_3

		$p_{1^3}/3!$	$p_{12}/2$	$p_3/3$
		$e_1^3/6$	$e_1^3/2 - e_1e_2$	$e_1^3/3 - e_1e_2 + e_3$
s_{1^3}	$e_1^3 - 2e_1e_2 + e_3$	1	1	1
s_3	e_3	1	-1	1
s_{12}	$e_1e_2 - e_3$	2	0	-1

The Schur polynomials are also the characters of the special inear groups. In fact the Weyl character formula expresses these characters as a quotient of two sums over the Weyl group. The Weyl group is the symmetric group, so the sums can be written as determinants, and turn out to be $a_{\lambda+\rho}$ and a_ρ for a suitable change in notation. The Scur functions are interpreted differently for the symmetric groups and the special linear groups: for symmetric groups the characters are given by regarding the Schur functions as linear combinations of Newton's symmetric polynomials (with the x_i being complex numbers of absolute value 1), while for the general linear group the Schur functions are regarded as functions on a maximal torus.