As we mentioned before, the Lie algebra cannot detect other components of the Lie group, or discrete normal subgroups. It is often useful to note that discrete normal subgroups of a connected group are always in the center (proof: the image of an element under conjugation is connected and discrete, so is just one point). So for example, O_n, SO_n and PO_n all have the same Lie algebra.

Subalgebras of Lie algebras B are defined in the obvious way and are analogues of subgroups. The analogues of normal subgroups are subalgebras A such that $[A, B] \subseteq A$ (corresponding to the fact that a subgroup is normal if and only, if $aba^{-1}b^{-1}$ is in A for all $a \in A$ and $b \in B$) so that B/A is a Lie algebra in the obvious way. Much of the terminology for groups is extended to Lie algebras in the most obvious way. For example a Lie algebra is called abelian if the bracket is always 0, and is called solvable if there is a chain of ideals with abelian quotients. Unfortunately the definition of simple Lie algebras and simple Lie groups is not completely standardized and is not consistent with the definition of a simple abstract group. The definition of a simple lie algebra has a trap: a Lie algebra is called simple if it has no ideals other than 0 and itself, and if the Lie algebra is NON-ABELIAN. In particular the 1-dimensional Lie algebra is NOT usually considered to be simple. The definition of simple is sometimes modified slightly for Lie groups: a connected Lie group is called simple if its Lie algebra is simple. This corresponds to the Lie group being non-abelian having no normal subgroups other than itself and discrete subgroups of its center, so for example $SL_2(\mathbb{R})$ is considered to be a simple Lie group even though it is not simple as an abstract group.

Subgroups of Lie groups are closely related to subgroups of the Lie algebra. Informally, we can get a subalgebra by taking the tangent space of the subgroup, and can get a subgroup by taking the elements "generated" by the infinitesimal group elements of the Lie algebra. However it is rather tricky to make this rigorous, as the following examples show.

Example 28 The rational numbers and the integers are subgroup of the reals, but these do not correspond to subgroups of the Lie algebra of the reals. It is clear why not: the integers are not connected so cannot be detected by looking near the identity, and the rationals are not closed.

This suggests that subalgebras should correspond to closed connected subgroups, but this fails for more subtle reasons:

Example 29 Consider the compact abelian group $G = \mathbb{R}^2/\mathbb{Z}^2$, a 2-dimensional torus. For any element (a, b) of its Lie algebra R^2 we get a homomorphism of $\mathbb{R}$ to G, whose image is a subgroup. If the ratio a/b is rational we get a closed subgroup isomorphic to S^1 as the image. However if the ratio is irrational the image is a copy of $\mathbb{R}$ that is dense in G, and in particular is not a closed subgroup. You might think this problem has something to do with the fact that G is not somply connected, because it disappears if we replace G by its universal cover. However G is a subgroup of the simple connected group $SU(3)$ so we run into exactly the same problem even for simply connected compact groups.

In infinite dimensions the correspondence between subgroups and subalgebras is even more subtle, as the following examples show.

Example 30 Let us try to find a Lie algebra of the unitary group in infinite dimensions. One possible choice is the Lie algebra of bounded skew Hermitian operators. This is a perfectly good Lie algebra, but its elements do not correspond to all the 1-parameter subgroups of the unitary group. We recall from Hilbert space theory that 1-parameter subgroups of the unitary group corespond to UNBOUNDED skew Hermitian operators (typical example: translation on $L^2(\mathbb{R})$ corresponds to d/dx, which is not defined everywhere.) So if we define the Lie algebra like this, then 1-dimensional subgroups need not correspond to 1-dimensional subalgebras. So instead we might try to define the Lie algebra to be unbounded skew hermitian operators. But these are not even closed under addition, never mind the Lie bracket, because two such operators might have no non-zero vectors in their domains of definition.

Example 31 It is reasonable to regard the Lie algebra of smooth vector fields as something like the Lie algebra of diffeomorphisms of the manifold. However elements of this Lie algebra need not correspond to 1-parameter subgroups. To see what can go wrong, consider the vector field $x^2 d/dx$ on the real line. If we try to find the flow corresponding to this we have to solve $dx/dt = x^2$ with solution $x = x_0/(1 - x_0 t)$. However this blows up at finite time t. so we do not get a 1-parameter group of diffeomorphisms. A similar example is the vector field d/dx on the positive real line: it corresponds to translations, but translations are not diffeomorphisms of the positive line because you fall off the edge.

So in infinite dimensions 1-parameter subgroups need not correspond to 1-dimensional subalgebras of the Lie algebra, and 1-dimensional subalgebras need not correspond to 1-parameter subgroups.

3 The Poincaré-Birkhoff-Witt theorem

We defined the Lie algebra of a Lie group as the left-invariant normalized differential operators of order at most 1, and simply threw away the higher order operators. This turns out to lose no information, because we can reconstruct these higher order differential operators from the Lie algebra by taking the "universal enveloping algebra", which partly justifies the claim that the Lie algebra captures the Lie group locally.

The universal enveloping algebra Ug of a Lie algebra g is the associative algebra generated by the module g, with the relations $[A, B] = AB - BA$. A module over a Lie algebra g is a vector space together with a linear map f from g to operators on the space such that $f([a, b]) = f(a)f(b) - f(b)f(a)$.

Exercise 32 Show that modules over the algebra Ug are the same as modules over the Lie algebra g, and show that Ug is universal in the sense that any map from the Lie algebra to an associative algebra such that $[A, B] = AB - BA$ factors through it. (Category theorists would say that the universal enveloping algebra is a functor that is left adjoint to the functor taking an associative algebra to its underlying Lie algebra.)

The universal enveloping algebra of a Lie algebra can be thought of as the ring of all left invariant differential operators on the group (while the Lie algebra consists of the normalized ones of order at most 1). Actually this is only correct in characteristic 0: over fields of prime characteristic it breaks down because not all left invariant differential operators can be generated by left invariant vector fields. We can see this even in the case of the 1-dimensional abelian Lie algebra over the integers.

Exercise 33 Show that the space of translation-invariant differential operators on $Z[x]$ has a basis of elements $\frac{1}{n!}\frac{d^n}{dx^n}$

So these are definitely not generated by $\frac{d}{dx}$. If we reduce mod p we get similar problems over fields of characteristic p. This is really a sign that in characteristic $p > 0$ the Lie algebra is not the right object (and does NOT capture the Lie group locally): the correct replacement is the algebra of all left invariant differential operators, or something closely related such as a formal group.

We need some control over the size of the universal enveloping algebra. Suppose that g is a free module over a ring. It is easy to find a good upper bound on the size of Ug. We can filter Ug by $U_0 \subset U_1 \subset U_2 \cdots$ where U_n is spanned by monomials that are products of at most n elements of g.

Exercise 34 The associated graded ring $U_0 \oplus U_1/U_0 \oplus \cdots$ has the following properties: is is generated by U_1, and is commutative.

Commutativity follows using $AB - BA = [A, B]$. So this graded algebra is a quotient of the polynomial ring on g, which gives an upper bound on the size of Ug.

Theorem 35 *The Poincaré-Birkoff-Witt theorem says that the map from the polynomial ring $S(g)$ to this graded algebra $G(Ug)$ of $U(g)$ is an isomorphism, in other words there is no further collapse.*

In particular the map from the Lie algebra to the UEA is injective, or in other words we can find faithful representations of the Lie algebra.