32 Symmetric functions and representations of symmetric groups

Schur-Weyl duality gives a correspondence between representations of symmetric and general linear groups. So in order to understand representations of general linear groups we would like to know the representations of symmetric groups. We will describe these using symmetric functions.

32.1 The ring of symmetric functions

Recall that conjugacy classes of symmetric groups S_n correspond to partitions of n. The irreducible representations can also be indexed by partitions. (Although finite groups have the same number of conjugacy classes and irreducible representations, it is not in general true that there is a natural correspondence between them: symmetric groups are unusual in that they do have such a natural correspondence.) We will describe the representation theory in terms of symmetric functions. More precisely, the conjugacy classes of S_n will correspond to Newton's symmetric functions of degree n, irreducible representations of S_n will correspond to Schur polynomials of degree n, and the character table of S_n is just the matrix for expressing Schur functions as linear combinations of Newton's functions.

The symmetric functions of n variables $x_1, \ldots, x_n$ are the polynomials in the elementary symmetric functions $e_1 = \sum x_i$, $e_2 = \sum_{i<j} x_i x_j$, ..., $e_n = \prod x_i$. It is convenient to take a sort of limit as n tends to infinity and define the ring of symmetric functions to be polynomials in an infinite number of variables $e_1, e_2, \ldots$. The point is that formulas involving symmetric functions tend to be independent of the number of variables x_i provided this number is sufficiently large.

The ring of symmetric functions has a lot of structure:

- A commutative product
- A cocommutative coproduct
- An antipode (or involution)
- A partial ordered
- A symmetric bilinear form
- Several different natural bases

The ring of symmetric functions has several useful sets of generators and bases.

- The elementary symmetric functions $e_n = \sum_{i_1<i_2<\cdots<i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$ form a generating set. The symmetric functions e_λ for λ a partition form a base. We put $E(x) = \sum e_i x^i = \prod(1 + x_i x)$, so it is a power series that formally has roots $-1/x_i$.
- The complete symmetric functions $h_n = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$ form a generating set. We have $H(x) = \sum h_i x^i = \prod(1 - x x_i)^{-1} = 1/E(-x)$.

- Newton's symmetric functions $p_n = \sum_{i>0} x_i^n$ form a generating set over the rationals, but not over the integers. We have $P(x) = \sum_{n>0} p_n x^n = \sum_i x_i x/(1-x_i x) = x\frac{d}{dx}\log(H(x) = xH'(x)/H(x)$.
- The Schur functions s_λ (see later)
- The monomial functions m_λ.
- The forgotten monomial functions

Exercise 354 Show that $E(-x)P(x) = -xE'(-x)$ and use this to prove Newton's identities giving recursive formulas for the sums p_i of the powers of roots of a polynomial $x^n - e_1x^{n-1} + \cdots$ in terms of its coefficients.

This gives at least 6 natural bases for the vector space of symmetric functions. Mathematicians working on symmetric functions spend many happy hours expressing writing the various basis elements as linear combinations or polynomials of other basis elements.

The ring of symmetric functions has a bilinear form $\langle , \rangle$ defined by the property that the symmetric functions p_λ form an orthogonal base or norm z_λ where z_λ is the order of the centralizer of a permutation of shape λ. The reason for this will appear later: when homogeneous symmetric functions are identified with class functions, this inner product becomes the usual inner product of class functions.

Exercise 355 Show that if a permutation has shape $1^{n_1}2^{n_2}\cdots$ then $z_\lambda = 1^{n_1}n_1!2^{n_2}n_2!\cdots$.

Recall that if V is a finite dimensional vector space with a symmetric nondegenerate inner product, then $\sum a_i a_i' \in S^2V$ summed over a basis a_i (with a_i' the dual basis) is independent of the choice of basis. We would like to do this for the space Λ but run into the problem that Λ is infinite dimensional. This is easy to fix because Λ is graded with finite-dimensional piece, so we just use $\sum a_i a_i' t^{\deg a_i} \in S^2V[[t]]$ instead. This element is independent of the choice of homogeneous basis.

Lemma 356 *For any homogeneous basis of Λ, we have*

$$\sum a_i a_i' t^{\deg a_i} = \prod_{i,j}(1 - tx_iy_j)^{-1}$$

Proof We only need to check this for one choice of basis, since the left hand side is independent of the choice of basis. Of course we use the basis $a_i = p_\lambda$, $a_i' = p_\lambda/z_\lambda$. The right hand side is given by

$$\exp(\sum_{i,j}\sum_{n>0} t^n x_i^n y_j^n/n) = \exp(\sum_n t^n p_n(x)p_n(y)/n)$$

The coefficient of t^m on the right is

$$\sum_{|\lambda|=m} p_\lambda(x)\frac{p_\lambda(y)}{z_\lambda}$$

which proves the lemma as by definition p_λ/z_λ is a dual basis to p_λ. □

The ring of symmetric functions is a Hopf algebra. The Hopf algebra structure is defined by making $E(x) = \sum e_i x^i$ grouplike (with $e_0 = 1$), or in other words

$$\Delta(e_n) = \sum e_i \otimes e_{n-i}.$$

Exercise 357 Show that over the rationals, the primitive elements of this Hopf algebra are the linear combinations of p_i, and that the Hopf algebra is the universal enveloping algebra of the abelian Lie algebra spanned by the p_i.

We know that commutative Hopf algebras should be thought of as group schemes, so we can ask what the group scheme corresponding to the Hopf algebra of symmetric functions looks like.

Exercise 358 Show that if G is the group scheme corresponding to the ring of symmetric functions, then for a commutative ring R, $G(R)$ can be identified with the multiplicative group of power series with leading coefficient 1 and coefficients in R.

The antipode of this Hopf algebra is given by $e_n^* = (-1)^n h_n$. This is slightly different from the involution often used on the ring of symmetric functions taking e_n to h_n. The two involutions differ on homogeneous elements of degree n by a factor of $(-1)^n$.

The ring of symmetric functions also turns up in other areas of mathematics in different guises. Here are a few apparently unrelated objects all of which are really the same ring, or rather Hopf algebra.

- The ring of symmetric functions
- Representations of symmetric groups
- Representations of general linear groups
- The homology of BU, the classifying space of the infinite unitary group. (It also turns up in several other related generalized homology rings of spectra.)
- Cohomology of Grassmannians ("Schubert calculus")
- The universal commutative λ-ring on one generator e_1
- The coordinate ring of the group scheme of power series with leading coefficient 1 under multiplication
- The Hall algebra of finite abelian p-groups, specialized to $p = 1$.
- It is the underlying space of a bosonic vertex algebra on 1 variable.
- It is the ring of polynomial functors on vector spaces.