The Weyl character formula also has infinite dimensional generalizations that we will discuss briefly. It works for some highest weight representations of Kac–Moody algebras: these are constructed from an infinite root system in the same way that we constructed algebras from finite root systems. To prove the Weyl character formula we needed 3 ingredients: Verma modules, the Weyl group,and the Casimir element. Verma modules for infinite dimensional Lie algebras are much the same as for finite dimensional ones. To get an action of the Weyl group we used the fact that the representation was finite dimensional but we did not need the full force of this: all we really needed was that the representation is "integrable", in other words splits as a sum of finite dimensional representations for each $SL2$, which in turn follows if the highest weight has suitable inner product with each simple root. We can do the same if the Lie algebra is infinite dimensional. The Casimir element is more of a problem, since its definition for an infinite dimensional Lie algebra becomes a divergent infinite sum. Kac discovered how to renormalize it to make sense. We first use the form $2\sum e_i f_i + \sum h_i h_{i'} + \sum [e_i, f_i]$. All these terms except the last make sense on a highest weight vector, but $\sum [e_i, f_i]$ is still an infinite sum of elements of the Cartan subalgebra. it is formally a sort of Weyl vector. In finite dimensional the Weyl vector can also be defined by $(\rho, \alpha) = (\alpha, \alpha)/2$ for all simple roots α, and this definition makes sense in infinite dimensions. Using this modified definition of the Casimir element, Kac showed that it behaves like the Casimir element for finite dimensional Lie algebras and in particular commutes with all elements of the Lie algebra. The proof of the Weyl–Kac character formula for integrable highest weight modules now goes through much as in the finite dimensional case.

Example 352 The Weyl denominator formula for the affine A_1 Kac-Moody algebra is the Jacobi triple product identity

$$\prod (1 - zq^{2n-1})(1 - z^{-1}q^{2n-1})(1 - q^n) = \sum_n (-1)^n z^n q^{n^2}$$

The product is over positive roots of the affine algebra $SL_2[t, t^{-1}]$, and the sum on the right hand side is really a sum over its Weyl group, which is the infinite dihedral group. Not all roots are conjugate under the Weyl group to simple roots. The infinite dihedral group is almost the integers, so sums over it can be written as sums over the integers. More generally there is a similar identity for every finite dimensional simple Lie algebra (or super algebra); these are the Macdonald identities.

Example 353 Suppose that $j(\tau) - 744 = q^{-1} + 196884q + \cdots = \sum c(n)q^n$ is the elliptic modular function. Then

$$j(\sigma) - j(\tau) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}$$

is a denominator formula for an infinite dimensional Lie algebra of rank 2. Here $q = e^{2\pi i\tau}$, $p = e^{2\pi i\sigma}$, and the root multiplicities of the root $(m, n) \neq (0, 0)$ of the Lie algebra is $c(mn)$. The Weyl group has order 2, and p^{-1} is really the term coming from the Weyl vector.

Alternatively one can stick to finite dimensional Lie algebras, but ask for the characters of infinite dimensional irreducible quotients of Verma modules. As in the finite dimensional case, we can take a resolution by Verma modules. The main problem is that we no longer have an action by the Weyl group. We can still write the character as

$$\sum_{w\in W} \frac{c(w)e^{w(\lambda+\rho)}}{e^{\rho}\prod_{\alpha>0}(1-e^{-\alpha})}$$

by using all the higher Casimirs, which together show that a Verma module with highest weight μ only occurs in a minimal resolution if $\lambda+\rho$ is conjugate to $\mu+\rho$. However the coefficients $c(w)$ need no longer be ± 1 as we do not have antiinvariance under the Weyl group to determine them. They can be quite complicated, and are given by values of Kazhdan–Lusztig polynomials (by the Kazhdan-Lusztig conjecture, now proved).

31 Jacobi triple product identity

The denominator formula for the affine A_1 algebra is the Jacobi triple product identity

$$\prod(1-zq^{2n-1})(1-z^{-1}q^{2n-1})(1-q^{2n}) = \sum_n (-1)^n z^n q^{n^2}$$

This function (and its generalizations for other simple groups) turns up in several other places:

- It is the denominator formula of a Kac-Moody algebra
- It is the simplest example of a Macdonald identity: an infinite product identity associated to a simple Lie algebra, or more generally an affine root system.
- It is closely related to the Boson-Fermion correspondence
- It is an example of a theta function: these can be thought of as either functions that are periodic up to elementary exponential factors, or as sections of line bundles over elliptic curves (or abelian varieties in the higher dimensional case)
- It is a solution of the heat equation; in fact it is essentially the fundamental solution to the heat equation on a circle.
- By restricting to special values of z one gets modular forms.
- It is the archetypal example of a Jacobi form.
- It spans a representation of finite Heisenberg groups. This does not seem very interesting as it is only the 1-dimensional trivial representation, but by taking slightly more general functions one gets higher dimensional representations.

We start by explaining the relation with the Boson-Fermion correspondence. We rewrite it as

$$\prod(1+zq^{n-1/2})(1+z^{-1}q^{n-1/2}) = \sum_m z^m q^{m^2} \prod(1-q^n)^{-1}$$

and find a combinatorial interpretation of both sides. We suppose that we have some fermions that can occupy energy levels $n+1/2$ for any integer n, with at most one fermion occupying each energy level. To avoid having negative energy states we fill up all the negative energy states with a Dirac sea, and take this to be our vacuum. Then we can either add fermions of energy $1/2, 3/2, ...$ or we can add antiparticles of energy $1/2, 3/2, ...$ by removing the particles of energy $1/2, 3/2, ..$ from the vacuum. The total energy and particle number are defined in the obvious way, by decreeing that the vacuum has particle number 0 and energy 0, and all other states will be obtained by adding a finite number of fermions of particle number 1 and their antiparticles of particle number -1. Then the total number of states of energy E and particle number N is the coefficient of $z^N q^E$ in

$$\prod(1+zq^{n-1/2})(1+z^{-1}q^{n-1/2})$$

Now we count this number in a different way. First do the case of particle number 0. We can obtain any state by of particle number 0 and energy E by starting with the vacuum and lifting the fermions to higher energy states; think of hitting them with photons of integral energy, each of which lifts one boson to a higher state. The number of ways to do this is the number of partitions of E, which is the coefficient of q^E in

$$\prod(1-q^n)^{-1}$$

So this verifies that the coefficient of z^0 in both sides is the same. What about other powers of z, in other words other particle numbers? For this we use the vacuum for particle number N, where we will the states of energy $1/2, 3/2, ..., N-1/2$, which has total energy N^2. So we just have to include extra terms $z^N q^{N^2}$ for all integers N to account for the vacuums of non-zero particle number.

The Boson-fermion correspondence comes from two different ways of looking at these sets. We can take the sets as basis elements for vector spaces. Then on the one hand we have an exterior algebra of an infinite dimensional space (which is what you get by quantizing a space of fermions) and on the other side we get a tensor product of a symmetric algebra (from quantizing bosons) tensors with a direct sum of vacuum states.

Notice that for finite exterior algebras is does not really matter which state you start with, but in infinite dimensions you get fundamentally different spaces depending on what you take as the vacuum: for example, some spaces have all states of positive energy, while others have states of arbitrarily large negative energy. This is a well-known problem in quantum field theory, that spaces you might expect to be isomorphic from analogy with finite dimensional spaces in quantum mechanics turn out to be fundamentally different.