For higher rank groups the Weyl character formula becomes rather unwieldy for practical calculations because the sums over the Weyl group become large. In this case a variation of it called the Freudenthal multiplicity formula is easier to use. This states

$$(\Lambda+\rho)^2-(\lambda+\rho)^2\mathrm{Mult}(\lambda)=2\sum_{\alpha}>0\sum_{j>0}(\lambda+j\alpha,\alpha)\mathrm{Mult}(\lambda+j\alpha)$$

where the sums are over positive roots α and positive integers j. Moreover the sum over the positive roots can usually be reduced in size by grouping things into orbits under the Weyl group. The Freudenthal formula can be proved by calculating the trace of the Casimir element on a weight space in two different ways. On the one had the trace is given by the dimension of the roots space times the eigenvalue $\Lambda^2+2(\Lambda,\rho)$. On the other hand the trace of each term e_if_i in the Casimir can be calculated explicitly by decomposing the representation into a sum of SL_2 modules and working out the trace on the weight spaces each of these; this gives terms depending on the number of times each SL_2 module occurs, which is in turn determined by the multiplicities of the weights $\lambda+j\alpha$ for various α. Finally the terms $h_ih_{i'}$ and ρ in the Casimir operator are easy to deal with. Putting everything together gives the Freudenthal recursion formula.

Example 345 We will demonstrate that this really does give a practical method by using it to calculate the character and dimension of a representation of E_8 with highest weight of norm 4. There are 3 possible orbits of weights under the Weyl group of norms 4 (2160 vectors), 2 (240 vectors), and 0 (1 vector). The vectors or norm 4 all have multiplicity 1 as they are conjugate to the highest weight vector. (Note that the Freudenthal formula give no information in this case.) We need to know the Weyl vector of E_8. Taking the simple roots to be $(...,1,-1,...)$ and $(-1/2,-1/2,-1/2,1/2,1/2,1/2,1/2,1/2)$ gives Weyl vector $(12,11,10,9,8,7,6,5)$ of norm $\rho^2=620$. The norm 4 vector in the Weyl chamber is $\Lambda=(3/2,(1/2)^7)$ with $(\Lambda,\rho)=46$. If $\lambda=((1/2)^7,-1/2)$ is the norm 2 vector in the Weyl chamber then $(\lambda,\rho)=29$. There are 63 positive roots having inner product 0 with λ. So the Freudenthal formula becomes

$$(4+2\times 46-2-2\times 29)\mathrm{Mult}(\lambda)=2\times 2\times 63$$

so the multiplicity of λ is 7.

Similarly if $\lambda=0$ we find

$$(4+2\times 46-0-2\times 0)\mathrm{Mult}(\lambda)=2\times 2\times 120\times 7$$

so the multiplicity of 0 is 35. The dimension of this representation is therefore $2160\times 1+240\times 7+1\times 35=3875$.

Notice that for the Weyl character formula the large size of the Weyl group was a disadvantage because we had to sum over it, but for the Freudenthal formula the large size is an advantage, because it means there are only a few orbits we have to sum over.

Exercise 346 Find the character of the irreducible representation with a highest weight vector of norm 6. Find the character of the alternating square of the adjoint representation and find out how it splits into irreducible representations.

Example 347 We can also work out the character values of characters of E_8 on elements of finite order. To show how to do this we will do the simplest case of the character of the adjoint representation on elements of order 2. The elements of order dividing 2 correspond to orbits of the Weyl group on $L/2L$ and we saw there are 3 orbits, represented by half of vectors of norms 0, 2, 4. If β is a vector of norm 2, then the order 2 element corresponding to $\beta/2$ acts as 1 on weight spaces if the weight has even inner product with β and -1 if the weight has odd inner product with β. We saw earlier that there are $1+56+134+56+1$ weights having inner products 2, 1, 0,-1, -2, so the trace of the order 2 element is $1-56+134-56+1=24$. Similarly for the norm 4 vector the corresponding order 2 element has trace $14-64+92-64+14=-8$.

We can use these two elements of order 2 to find the two non-compact real forms of $E8$ by the usual construction: take the points of $\mathbb{C}\otimes E8$ fixed by an antilinear involution. One of these real forms is the split form, with fixed point subalgebra D_8, and the other is a new real form of E_8 with fixed point subalgebra $E7\times SL2$. So we have found the 4 simple Lie algebras associated with E_8: a complex form (of real dimension 2×248), a compact form, a split form, and another form.

Example 348 What are the maximal compact subgroups of the two non-compact real forms? Their Lie algebras are D_8 and $E_7\times SL_2$, but there are several different connected groups with these Lie algebras as their centers are different. For the one with compact subalgebra D_8, we look at the action on the Lie algebra of E_8 and see that it splits as (adjoint representation) plus (half spin representation). So the corresponding compact subgroup is the image of the spin group under this representation, which is the quotient of the sing group by an element of order 2 in the center, different from the one giving the special orthogonal group.

Exercise 349 Similarly show that the maximal compact subgroup of the other real form is the quotient of $E7\times SL_2(\mathbb{R})$ by the element $(-1,-1)$ of the center.

Our method of finding the elements of order 2 in $E8$ was somewhat ad hoc. There is a more systematic way of finding all elements of finite order as follows. To find elements of order dividing n, We want to classify the elements of $\frac{1}{N}L/L$ up to conjugacy by the Weyl group, where L is the $E8$ lattice, as these correspond to orbits of elements of order dividing N in a maximal torus. In particular we can assume such an element is in the Weyl chamber, so has non-negative inner product with all simple roots. Moreover we can assume that 0 is the nearest lattice point. This implies that $(v,\alpha)\leq(\alpha,\alpha)/2=1$ where α is the highest root (the root in the Weyl chamber) otherwise this root would be closer. So if the inner products with the simple roots are n_i/N, then $\sum m_in_i/N\leq 1$ where the m_i are the numbers giving the highest root as a linear combination of the simple roots. This can be written more neatly as $\sum m_in_i=N$ where we now include $m_0=1$ in the sum. This makes it easy to list the conjugacy classes of elements of any given order in E_8. For example, there are 5 classes of order dividing 3, so 4 classes of order 3.

Exercise 350 Find the conjugacy classes of $E8$ of elements of order at most 6.

Example 351 There is an amusing way to calculate the order of the Weyl group of $E8$ by looking at the number of elements of order dividing N for large N. On the one hand, this number is about N^8 times the index of the root lattice in the weight lattice divided by the order of the Weyl group. On the other hand it is the coefficient of x^N in $\prod(1-x_{m_i})^{-1}$, which is about $N^8/8!\prod m_i$. SO the order of the Weyl group is $8!\times 1\times 2\times 3\times 4\times 5\times 6\times 4\times 3\times 2$. A similar method works for other Lie algebras.