To do this we use the Casimir element. Recall that we have a symmetric invariant bilinear on the Lie algebra, which gives us an element of the center of the UEA of the form $\sum a_i a_i\prime$ where a_i forms a basis and $a_i\prime$ is the dual basis. In our case the Casimir is given by $\sum_\alpha e_\alpha e_{-\alpha} + \sum h_i h_i\prime$. We want to apply this to a highest weight vector of a Verma module, so we move all the vectors of negative root spaces over to the left, picking up a term $[e_{-\alpha}, e_\alpha]$ for each positive root α. So the eigenvalue on a highest weight vector with highest weight λ is $\lambda^2 + (\lambda, 2\rho) = (\lambda+\rho)^2 - \rho^2$. In particular two Verma modules have the same eigenvalue of the Casimir only if they have the same value of $(\lambda+\rho)^2$.

This is enough to complete the proof of the Weyl character formula, because any possible highest weight μ that is in the fundamental chamber of the Weyl group and is of the form $\lambda -$ sum of positive roots satisfies $(\mu+\rho)^2 < (\lambda+\rho)^2$ unless $\lambda = \mu$ by elementary geometry.

Example 342 We can work out the characters of representations of B_2 using the fact that the alternating sum over an octagon of the root multiplicities is usually zero. We can also decompose representations by writing them as a linear combination of irreducible characters, or more efficiently by observing that the multiplicity of highest weight vectors is given by an alternating sum over octagons. For example, $4 \times 10 = 20 \oplus 16 \oplus 4$.

For rank 2 groups A_1^2, A_2, B_2, G_2, we get the multiplicities of representations by looking at alternating sums over a square, hexagon, octagon, or dodecagon. This polygon is easy to remember, as it starts off with the origin and the two simple roots.

We can also work out the dimension of a representation from the Weyl character formula. Substituting in the identity element of the group direct fails badly, because both the numerator and the denominator have a zero of high order (half the number of roots) at the identity. In fact, from this point of view the identity element is the most complicated element of the group! (It is also the place where the set of unipotent elements has the worst singularity.) So instead we examine the asymptotic behavior of the numerator and denominator of the Weyl character formula on a carefully chosen set of elements that tend to the identity.

For this we use the Weyl denominator formulas

$$\sum_{w\in W} \epsilon(w) e^{w(\rho)} = e^{-\rho} \prod_{\alpha>0} (1 - e^\alpha)$$

which gives

$$\sum_{w\in W} \epsilon(w) e^{(w(\rho), t\beta)} = \prod_{\alpha>0} (e^{(\alpha/2, t\beta)} - e^{(-\alpha/2, t\beta)})$$

for any real t and any β in the dual of the Cartan subalgebra. We rewrite this as

$$\sum_{w\in W} \epsilon(w) e^{(w(\beta), t\rho)} = \prod_{\alpha>0} t \frac{(e^{(\alpha/2, t\beta)} - e^{(-\alpha/2, t\beta)})}{t}$$

and examine the behavior as t tens to 0. The right hand side behaves like

$$t^N \prod_{\alpha>0} (\alpha, \beta) + \text{higher powers of} t$$

so this gives the asymptotic behavior of the numerator and denominator of the Weyl character formula on the elements $t\rho$. We can now cancel out the factors of t^N and take the limit as t tends to 0, to find that the dimension is

$$\prod_{\alpha>0} \frac{\lambda+\rho,\alpha)}{\rho,\alpha}$$

Example 343 We check the Weyl dimension formula on G_2. If α and β are the long and short simple roots of norms 6 and 2 then the positive roots are $\beta, \alpha, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta$. Suppose the highest weight *lambda* has inner product $3m$ with α and n with β, with m and n non-negative integers. Then the dimension is

$$\frac{n+1}{1} \times \frac{3m+3}{3} \times \frac{3m+n+4}{4} \times \frac{3m+2n+5}{5} \times \frac{3m+3n+6}{6} \times \frac{6m+3n+9}{9}$$

For $m=0$, $n=2$ this gives the dimension 27 that we found earlier.

Example 344 If we take $n\rho$ as a highest weight, we see that there are irreducible representations of dimension n^N where N is the number of positive roots. This is related to the Steinberg representation of a finite group of Lie type over a field of order q, an irreducible representation of dimension q^N.