Theorem 335 *If G is a connected quasi-simple compact Lie group, there is an element q such that an irreducible representation is real or quaternionic depending on whether q acts as $+1$ or -1.*

Proof For the group SU_2 we can check this by direct calculation: the irreducible representations of even dimension have an alternating form, and those of odd dimension have an even form. So the element q is the non-trivial element of the center.

The idea of the proof for general compact groups is to reduce to this case by finding a homomorphism from SU_2 to G such that the restriction of any irreducible representation V of G to this SU_2 contains some irreducible representation W with multiplicity exactly 1. Then any alternating or symmetric form on V must restrict to an alternating or symmetric form on W, so we can tell which by examining the action on W of the image $q \in G$ of the nontrivial element of the center of the SU_2 subgroup.

We will find a suitable SU_2 subgroup by constructing a basis E, F, H for its complexified Lie algebra. We take E to be the sum $\sum_\alpha E_\alpha$ where the sum is over the simple roots and E_α is some nonzero element of the simple root space of α. We take H to be the element of the Cartan subalgebra that has inner product 2 with every simple root, which is possible as the simple roots are linearly independent. Finally we choose F_α in the root space of $-\alpha$ so that $\sum [E_\alpha, F_\alpha] = H$. Then we see that $[H, E] = 2E$. $[E, F] = H$, and $[H, F] = -H$, so we have found an $\mathfrak{sl}_2$ subalgebra.

This subalgebra has the property that $(H, \alpha) > 0$ for every simple root α, which implies that if β is the highest weight of the representation V, then β restricted to $\langle E, F, H\rangle$ has multiplicity 1. So this subalgebra has the desired property that the restriction of any irreducible representation of G has some irreducible representation of multiplicity 1 $\square$

The $\mathfrak{sl}_2$ subalgebra constructed above is sometimes called a principal $\mathfrak{sl}_2$ subalgebra.

Exercise 336 If the dual Weyl vector ρ' has inner product 1 with all simple roots, show that a self-dual irreducible representation with highest weight α is real or quaternionic depending on whether it has even or odd inner product with ρ'. (The dual Weyl vector is essentially the Weyl vector of the "dual" root system, whose roots are the coroots of the root system, and is half the sum of the positive coroots. It can be identified with the Weyl vector if all roots have norm 2. It is essentially the same as the element H in the theorem above.)

Exercise 337 Find suitable elements E, F, H of $M_n(\mathbb{C})$ spanning a principal $\mathfrak{sl}_2$ subalgebra for the case when G is SU_n.

We can figure out this special element of order 2 for the various compact simply connected quasi-simple Lie groups as follows.

A_n q is the element of order 2 if $n \equiv 1 \bmod 4$, otherwise q is the identity.

B_n q is the element of order 2 if $n \equiv 1, 2 \bmod 4$, otherwise q is the identity.

C_n The $2n$-dimensional representation has an alternating bilinear form, so q must be the non-trivial element of the center.

D_n q is an element of order 2 if $n \equiv 1, 2 \bmod 4$ otherwise q is the identity.

E_7 The 56-dimensional representation has an alternating bilinear form (coming from the Lie bracket on $E_8 = E_7 \oplus 56 \otimes 2 \oplus sl_2$), so q must be the non-trivial element of the center.

G_2, F_4, E_6, E_8 The center has no elements of order 2, so q is the identity.

Exercise 338 Which compact simply connected Lie groups have the property that all representations have a symmetric invariant bilinear form?

For the compact connected Lie groups, for any quaternionic representation there is an element of order 2 in the center acting as -1. This is no longer true for quaternionic representations of finite simple groups. There are some (rare) examples of simple finite groups (with no center) with irreducible quaternionic representations. For sporadic finite simple groups real representations are most common and quaternionic representations are quite rare.

30 Weyl character formula

For irreducible representations of general linear groups, we have seen that the characters are given by Schur functions, which can be written as a quotient of two alternating sums over the Weyl group. The Weyl character formula is a generalization of this to all semisimple finite-dimensional complex Lie algebras. It also works for the corresponding complex Lie groups, compact Lie groups, and so on.

Theorem 339 *Weyl character formula. If V is an irreducible representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight λ, then the character of V is given by*

$$\frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho}}{e^{\rho} \prod_{\alpha>0}(1-e^{-\alpha})}$$

The special case of the trivial representation with highest weight 0 is the Weyl denominator formula:

$$\sum_{w \in W} \epsilon(w) e^{w(\rho} = e^{-\rho} \prod_{\alpha>0} (1-e^{\alpha})$$

In the case of the general linear group, this is just the Vandermonde identity, as the left hand side is the expansion of the Vandermonde determinant, and the right hand side is its expression as a product. We can use the denominator identity to replace the denominator in the Weyl denominator formula by an alternating sum over the Weyl group.

Example 340 For the group $SL_2(\mathbb{C})$, the Weyl group has order 2, and the Weyl character formula says that the characters are given by

$$\frac{q^{n+1} - q^{-n-1}}{q^{-1}(1-q^2)} = q^{-n} + q^{2-n} + \cdots + q^{n-2} + q^n$$

There are several different ways to prove the Weyl character formula:

- Weyl's original method used the Weyl integration formula for compact Lie groups. The denominator appears as the square root of the weight in the Weyl integration formula.
- It can be proved using a resolution of the irreducible representation by Verma modules. In this case the denominator of the Weyl character formula is the character of a Verma module, and the sum in the numerator is a sum over the Verma modules appearing in the resolution. Kac extended this proof to Kac-Moody algebras.
- It can be proved by studying the Lie algebra homology of the nilradical $\mathfrak{n}$ of a Borel subalgebra. The denominator is then the character of the universal enveloping algebra of $\mathfrak{n}$, and the sum in the numerator is a sum over a basis for the homology groups of $\mathfrak{n}$.

Example 341 We calculate the characters of some representations of G_2. We have already found the first few, of dimensions 1, 7, 14. The next has highest weight twice a shortest root. To use the Weyl character formula, observe that it gives a recursive relation for the character multiplicities, because the product of the character and $\sum \epsilon(w)e^{w(\rho)}$ is usually 0. This means that the alternating sum of the multiplicities over the "dodecahedrons" coming from conjugates of the Weyl vector are 0, unless the dodecahedron passes through exactly one weight. Using this we see that the multiplicities of the vectors are 1 (2 times shortest root) 1 (longest root) 2 (shortest root) and 3 (0), for a total dimension of 27.

We will sketch a proof using Verma modules. Recall that we constructed the irreducible representation of a quotient of a Verma module by other Verma modules. We know the character of a Verma module, so we can try to find the character of the irreducible module by subtracting the characters of the Verma modules we quotient out by:

$$\frac{e^{\lambda} - \sum_{\alpha \text{simple}} e^{\lambda-(2(\lambda,\alpha)/(\alpha,\alpha)+1)\alpha}}{\prod_{\alpha>0}(1-e^{-\alpha})}$$

This gives a first approximation to the character, but is obviously wrong if the Verma modules we quotient out by overlap. In this case we have to add back in terms for their intersections, then subtract further terms when three intersect, and so on. More precisely, we can write down a resolution of the irreducible module by Verma modules, and the character will be an alternating sum of the characters of the terms of the resolution. So far we have only found the first two terms of the resolution.

To see how to fix this, first look at the denominator. If we rewrite it as

$$e^{-\rho}\prod_{\alpha>0}(e^{\alpha}/2 - e^{-\alpha/2})$$

we see that it is alternating under the Weyl group apart from the factor of $e^{-\rho}$ where the Weyl vector ρ is half the sum of the positive roots. Since the character is invariant under the Weyl group, the numerator must be antiinvariant under the Weyl group, at least if we multiply by e^{ρ}. The simplest possibility for the numerator is then

$$\sum_{w\in W} \epsilon(w)e^{w(\rho+\lambda}$$

where ϵ is the determinant of an element of the Weyl group. The first few terms of this sum for w trivial or a reflection are the terms corresponding to the Verma module we started with and the ones we quotiented out by. The problem is that we need to rule out the possibility of having other antiinvariant terms in the numerator such as

$$\sum_{w \in W} \epsilon(w) e^{w(\rho+\mu}$$

for some other vector μ. In other words we need to show that a minimal resolution of the irreducible module with highest weight λ only contains Verma modules with highest weights of the form $w(\lambda + \rho) - \rho$.