28.3 Compact Lie algebras

The Lie algebras we have constructed are defined over the reals, but are the split rather than the compact forms. We can get the compact forms by twisting them. If we have any involution ω of a real Lie algebra L, we can construct a new Lie algebra as the fixed points of ω extended as an antilinear involution to $L \otimes \mathbb{C}$. This is using the fact that real forms of a complex vector space correspond to antilinear involutions. The reason why we use involutions rather than elements of some other order is related to the fact that the Galois group of $\mathbb{C}/\mathbb{R}$ has order 2. More generally if we wanted to classify (say) rational Lie algebras, we would get a problem involving non-abelian Galois cohomology groups of the Galois group of the rationals.

Example 332 The Lie algebra of the unitary group is given by the complex matrices A with $A = -\overline{A}^T$, so the involution in this case is $A \mapsto -A^T$.

It is obvious how to generalize this example to the Lie algebras constructed from the Serre relations: the analogue of the transpose swaps e_i and f_i, so the involution ω takes e_i to $-f_i$, f_i to $-e_i$, and h_i to $-h_i$.

To show that the corresponding Lie group is compact, it is enough to find an invariant symmetric bilinear form on the Lie algebra that is definite, as the Lie group is then a subgroup of the orthogonal group. The split algebra has a symmetric bilinear form because the adjoint representation is self dual (minus a root is still a root) and this bilinear form can be normalized so that $(e_i, f_i) = 1$. This bilinear form is not definite: for example $(e_i, e_i) = 0$, but a straightforward calculation shows that its twist by ω is negative definite.

An immediate consequence is that the finite dimensional representations of all the simple Lie algebras we have constructed are completely reducible, by Weyl's unitarian trick: their finite dimensional representations are the same as those of a compact group.

For a compact Lie algebra, there may be several different connected compact Lie groups with different centers. We would like to find the simply connected group (and check it is compact!) so that all others are quotients of it. The key point is that the center of the simply connected compact group can be identified with the dual of the finite group (weight lattice/root lattice).

To see this, pick a finite dimensional representation and look at a Cartan subgroup of the group acting on this representation. Its image is isomorphic to (Cartan subalgebra/dual of lattice generated by the weights of the representation). Except for trivial representations, this lies between (Cartan subalgebra/dual of weight lattice) and (Cartan subalgebra/dual of root lattice). So we see that the Cartan subgroup of the simply connected group is (Cartan subalgebra/dual of root lattice) and its center is (Dual of root lattice/dual of weight lattice), which is the dual of (Weight lattice/root lattice). (Here we use the fact that for a compact group we can finite faithful finite dimensional representations, and in particular can detect anything in the center using finite dimensional representations.)

Actually we need a slight further argument: we have really only found the maximal finite cover of a compact group. However the fundamental group of the universal cover is finitely generated, so if their is a finite bound for the finite covers then the maximal finite cover is the universal cover, as any finitely generated abelian group such that there is a bound on the sizes of its finite quotients must be finite.

So now we can list the compact simple simply connected Lie groups:

- A_n: SU(n+1), center cyclic of order $n+1$.
- B_n: Spin double cover of $SO(2n+1)$: center cyclic of order 2.
- C_n: $Sp_{2n}(\mathbb{C}) \cap U(2n)$: center cyclic of order 2.
- D_n: spin double cover of SO_{2n}: center of order 4, cyclic if n is odd.
- E_6: Center order 3, faithful action on the 27-dimensional representations
- E_7: Center order 2, faithful action on the 56-dimensional representations
- E_8, F_4, G_2: center order 1, faithful action on adjoint representation.

While the group (weight lattice/root lattice) detects elements of the center that act nontrivially in finite dimensional representations, it fails to detect elements that act trivially in all finite dimensional representations. This cannot happen for compact groups, but is common for noncompact groups. We have already seen an example of this for $SL_2(\mathbb{R})$, whose universal cover has an infinite cyclic center, which is not equal to (weight lattice/root lattice). However the calculation of the center in the non-compact case can be reduced to the compact case, because for a real algebraic group the fundamental group is the same as that of its maximal compact subgroup.

29 Schur indicator for compact Lie groups

In this section all representations will be finite dimensional and groups will be compact Lie groups.

Theorem 333 *For a compact simply connected Lie group, every irreducible representation is self dual if and only if the Weyl group contains -1.*

Proof If α is the highest weight of an irreducible representation, then the lowest weight of its dual is $-\alpha$. There is a unique element w of the Weyl group taking the fundamental Weyl chamber W to $-W$, so $-w$ takes α to the highest weight of its dual. So every representation is self dual if and only if $-w = 1$, in other words if W contains -1. □

Exercise 334 The element $-w$ is an automorphism of the Dynkin diagram. Show that it is the nontrivial automorphism for diagrams of types A_n ($n \geq 2$), D_n (n odd), and E_6, and the identity for all other connected Dynkin diagrams. Show that an irreducible representation is self dual if and only if its highest weight is fixed by $-w$.

Next we have to figure out which of the self dual representations are real and which are quaternionic, or in other words find out whether their invariant bilinear forms are symmetric or alternating.