Conversely if α satisfies these conditions then we get a finite dimensional module by killing off all the elements $f^{1+2(\alpha,\alpha_i)/(\alpha_i,\alpha_i)}v$. The proof of this is similar to the proof that the Lie algebras were finite dimensional: the extra relations imply that the highest weight vector is contained in finite dimensional representations of all the sl_2's, which implies that every vector of the representation is contained in finite dimensional sl_2 modules. As before this implies that the representations of the Lie algebras sl_2 lift to representations of the groups SL_2, and therefore the Weyl group acts on the weight spaces. Just as for the case of Lie algebras, this implies that the representation is finite dimensional.

So this gives a complete list of the finite dimensional irreducible representations, though it is not obvious what their dimensions or characters are.

We should figure out what the weight lattices are. The weight lattices all contain the root lattices with finite index, $n+1$ for A_n, 2 for B_n, C_n, 4 for D_n, (though there is a difference for n even or odd, as the quotient may be a Klein 4-group or cyclic), 1 for E_8, F_4, G_2, 2 for E_7, and 3 for E_6.

In general it is not yet clear what the characters or dimensions are: this will later be given by the Weyl character and dimension formulas. However there are some easy cases where we can already write down the answers, where the representation is either minuscule or a representation of SL_3.

28.1 Minuscule representations.

There is one case where it is easy to work out the character of a representation: this is when all weights are conjugate under the Weyl group, so they all have the same multiplicity as the highest weight, which is of course 1. These are called minuscule representations. They tend to turn up a lot as the "smallest" representations of a group: in fact, together with the adjoint representation and the representation with highest weight a "short" root, they account for most of the finite-dimensional representations that people study explicitly.

Minuscule representations correspond to characters of the center of the simply connected compact group. More precisely, for each coset of the weight lattice, pick a vector of smallest norm. This is the highest weight of a minuscule representation.

Some authors do not count the identity representation as a minuscule representation, which is rather like removing the zero element from a vector space.

We can list the minuscule representations as follows:

- A_n: the minuscule representations are the exterior powers of the vector representation.
- B_n: the minuscule representations are the trivial representation and the splin representation. (The vector representation is not minuscule!)
- C_n: the minuscule representations are the trivial one and the vector representation.
- D_n: the minuscule representations are the trivial representation, the vector representation, and the two half spin representations.
- E_6: The minuscule representations are the trivial representation and the two 27-dimensional representations, which we constructed earlier by decomposing E_8

- E_7 The minuscule representations are the trivial representation and the 56 dimensional one.

28.2 Representations of SL3

For SL_3 we already have enough information to find the characters of the irreducible representations. The character of a Verma module is the character of a polynomial algebra on 3 generators, so looks like:

We can also work out the character of $V_\lambda/f^{1+(\lambda,\alpha_1)}$ as:

Using the fact that the character is invariant under the Weyl group, this gives the character of the irreducible representation: it is in the convex hull of a sort of semiregular hexagon, and the weight multiplicities increase by 1 every time one goes 1 step further in, until one gets to the "triangle" in the center when they become constant.

Example 329 The dimensions of the irreducible representations of dimension at most 30 are 1, 3, 3, 6, 6, 8, 10, 10, 15, 15, 15, 15, 21, 21, 24, 24, 27, 28, 28,... In general the dimension of the representation whose highest weight has inner products m, n with the simple roots is $(m+1)(n+1)(m+n+2)$: these factors are the inner products of $\lambda+\rho$ with the simple roots. This is a special case of the Weyl dimension formula.

Exercise 330 Verify that the irreducible representation of SL_3 whose highest weight has inner products m and n with the simple roots has dimension $(m+1)(n+1)(m+n+2)$.

Example 331 We can decompose tensor products of representations of SL_3 by calculating their characters and writing these as linear combinations of irreducible characters. For example, $6\otimes 6 = 15\oplus 15\oplus \overline{6}$, $6\otimes\overline{6} = 27\oplus 8\oplus 1$, $8\otimes 3 = 15\oplus 6\oplus\overline{3}$ and so on,

This graphical method is fine for small representations but becomes cumbersome for larger representations. The Littlewood-Richardson rule is a more efficient method for decomposing larger tensor products.