27 Hilbert's finiteness theorem

Given a Lie group acting linearly on a vector space V, a fundamental problem is to find the orbits of G on V, or in other words the quotient space. For example, one might want to find the binary forms of degree n up to equivalence under the action of SL_2. One way to attack this problem is to look at invariants: at least formally, the functions on the quotient space V/G might be the invariant functions $S(V^*)^G$ on V. There are a few problems with this as shown by the following examples:

Example 324 Suppose G is the group of non-zero reals acting on the reals, so there are two orbits. However the ring of invariants is just $\mathbb{R}$, so this does not show both orbits. The problem arises from the fact that one orbit is in the closure of the other, so any invariant function has the same value on both orbits and invariant functions cannot separate them. The quotient space in this case is not even Hausdorff. This problem does not appear if the group is compact, so all orbits are closed. In geometric invariant theory on deals with it by only considering "stable" or "semistable" orbits.

Example 325 Suppose G is the group of order 2 acing on the reals by -1. The ring of invariants is a polynomial ring on 1 variable, suggesting that the quotient space should be the real line. It is not: it is half the real line. What is happening is that there are some orbits $\{ix, -ix\}$ in $\mathbb{C}$ that are invariant under complex conjugation, even though the elements in the orbits are not. In other words the ring of invariants is not really detecting orbits of G on V, but rather orbits of G on $V \otimes C$ that are defined over the reals.

To summarize, we might expect the ring of invariants to tell us what the orbits are, provided the group is compact and its representation is complex. Otherwise the relation between the ring of invariants and the orbits is more subtle.

We can now ask if the space of orbits, or rather the spectrum of the ring of invariants, is an algebraic variety. It is an algebra over the complex numbers with no nilpotents, so it comes from an algebraic variety if and only if it is finitely generated. Hilbert proved that it was finitely generated in many cases.

We start by disposing of the standard myth about Hilbert's finiteness theorem. Gordan is supposed to have said about Hilbert's finiteness proof "this is not math; this is theology" as Hilbert's proof was not constructive. It is not all all clear if he really said this, since there is no written record of it until many years after he died, and in any case it may have been a joking compliment rather than a complaint, as Gordan thought highly of Hilbert's work.

Theorem 326 *If G is a Lie group whose finite dimensional representations are completely reducible, then the ring of invariants of G acting on a finite dimensional vector space is finitely generated.*

Proof We do the case when G is finite. A is graded by degree. Let I be ideal generated by positive degree elements of A^G. Then I is a finitely generated ideal by Hilbert basis theorem, with generators $i_1, \ldots, i_k$ which we can assume are fixed by G. We want to show that these generate A^G as an algebra, which is much stronger than saying they generate the ideal I. (Example: subring of

$k[x, y]$ generated by xy^* is NOT finitely generated, even though the corresponding ideal is. We need to use some special property of subrings fixed by a finite group.)

We use the Reynolds operator ρ given by taking average under action of G (which needs char=0, though in fact Hilbert's theorem is still true for finite groups in positive characteristic). Key properties: $\rho(ab) = a\rho(b)$ if a fixed by G, $\rho(1) = 1$. It is not true that $\rho(ab) = \rho(a)\rho(b)$ in general. ρ is a projection of A^G modules from A to A^G but is not a ring homomorphism.

We show by induction on degree of x that if $x \in A^G$ then it is in algebra generated by i's.

We know

$$x = a_1 i_1 + \ldots + a_k i_k$$

for some a's in A as x is in I. Apply Reynolds operator:

$$x = \rho(x) = \rho(a_1)i_1 + \ldots + \rho(a_k)i_k$$

By induction $\rho(a_j)$ is in A^G as it has degree less than that of x, so $x \in A^G$. $\square$

It is astonishing that one of the biggest problems in the 19th century can now be disposed of in a few lines of algebra. This is essentially Hilbert's proof, though his version of it occupied many pages. He had to develop background results that are now standard such as his finite basis theorem, and instead of using integration over compact groups used a more complicated operator called Cayley's omega process.

The simplicity of the proof may be a bit misleading: it is rather difficult in general to find explicit generators for rings of invariants, except for a few special cases such as reflection groups. The invariants tend to be horrendously complicated polynomials, and the number of them needed as generators can be enormous. In other words rings of invariants are usually too complicated to write down explicitly. In turn this suggests that the orbit space of a vector space under a Lie group may in general be rather complicated.

Compact groups: the proof is similar as can still integrate over the group:

Noncompact groups such as $SL_n(\mathbb{C})$: Use Weyl's unitarian trick: invariant vectors (for finite dimensional complex reps of the complex group) same as for compact subgroup SU_n, so still get Reynolds operator. Works for all semisimple or reductive algebraic groups (key point: reps are completely reducible), but NOT for some unipotent groups (Nagata counterexample to Hilbert conjecture). Char p harder as groups need not be completely reducible; e.g. Z/pZ acting on 2-dim space over $\mathbb{F}_p$. Haboush proved Mumford's conjecture giving a sort of nonlinear analogue of Reynolds operator, which can be used to prove finitely generated of invariants for reductive groups as in char 0.

Example 327 Classical invariant theory: $G = SL_2(\mathbb{C})$ acting on

$$a_n x^n + a_{n-1}x^{n-1}y + \ldots + a_0 y^n$$

, $A = \mathbb{C}[a_0, \ldots, a_n]$. A^G is the ring of invariants of binary forms, shown to be finitely generated by Gordan. More complicated examples in more variables shown to be finitely generated by Hilbert. Example of an invariant: the discriminant $b^2 - 4ac$ of $ax^2 + bxy + cy^2$.

Example 328 Hilbert asked if the finiteness theorem still holds for all groups, even if their finite dimensional representations are not completely reducible. Nagata found a counterexample as follows. Take the group R acting on R^2 by $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. The sum of 16 copies of this gives an action of R^{16} on a vector space R^{32}. Nagata showed that for a generic 13-dimensional subgroup G of the R^{16}, the ring of invariants is not finitely generated. The group G is just an abelian Lie group, showing again that abelian Lie groups are in some ways more complicated than the simple ones.

28 Finite dimensional representations of semisimple Lie groups

We now want to study the finite dimensional representations of the semisimple Lie groups we have constructed. There are several problems to solve:

- Find the irreducible representations.
- Find the dimension, and more generally the characters, of the irreducible representations
- Describe the tensor products, symmetric squares, and so on of representations. In other words find the structure of the lambda-ring generated by representations.
- Find natural geometric realizations of the representations.

We will solve the first problem, of parameterizing the irreducible representations, by showing that they correspond to weights in the fundamental Weyl chamber (Cartan's theorem).

If we have a representation of one of the Lie algebras we constructed from a finite root system, then we can look at eigenvectors of the Cartan subalgebra H. By hitting an eigenvector with elements e_i as much as possible, we can assume that the eigenvalue of the eigenvector is a highest weight, in other words the eigenvalue is killed by all the e_i and is an eigenvector of all the h_i. There is a universal module with these properties called a Verma module. We can construct it as an induced modules

$$U(E,H,F)\otimes_{U(E,H)} V$$

where V is any 1-dimensional module over H, where we let E act trivially on it, so is a module for the UEA $U(E,H)$ of the subalgebra generated by E and H. By the PBW theorem we can see that the Verma module can be identified with the UEA of F, so is the same size as the symmetric algebra on F.

The Verma module is of course infinite dimensional and we need to cut it down. Look at the action of one of the SL_2 subalgebras e_i, h_i, f_i on the highest weight vector v. This generates a Verma module for SL_2, which is infinite dimensional and usually irreducible. The only way to cut it down to something finite dimensional is to kill off $f^{1+2(\alpha,\alpha_i)/(\alpha_i,\alpha_i)}v$, where α is the weight of v and α_i is the root of e_i. So we get two necessary conditions on α for it to be the highest weight of a finite dimensional module:

1. (α, α_i) must be an integral multiple of $(\alpha_i, \alpha_i)/2$, in other words α must be in the "weight lattice"

2. (α, α_i) must be at least 0, in other words α must be in the fundamental Weyl chamber.