26 Invariants of reflection groups

The ring of invarinats of a Weyl group turns up a lot in representation theory. For example, Harish-Chandra showed that the center of the UEA of a semisimple Lie algebra (the "higher Casimir s") is isomorphic to the ring of invariants of the Weyl group. The rational homology of the compact Lie group can be read off from the ring of invariants of the Weyl group. Fortunately it turns out to have a very easy structure: it is a polynomial ring. (This is unusual: rings of invariants of finite groups are usually very complicated.)

For the reflection group S_n acting on R^n the invariants are just the symmetric functions, which form a polynomial ring generated by the usual elementary symmetric functions.

Exercise 313 Show that the ring of invariants of the reflection group $S_n = A_{n-1}$ acting on $\mathbb{R}^{n-1}$ (identified with the vector space of vectors in $\mathbb{R}^n$ whose sum of coordinates is 0) is the polynomial ring $\mathbb{C}[e_2, \cdots e_n]$ in $n-1$ generators.

Exercise 314 Show that the rings of invariants of the reflection groups B_n and D_n are polynomial rings, and find sets of generators.

Exercise 315 Show that the ring of invariants of the alternating group acting on $x_1, \cdots, x_n$ is generated by the invariants $\mathbb{C}[e_1, \ldots, e_n]$ of the symmetric group together with the invariant $\Delta = \prod_{i<j}(x_i - x_j)$. Show that Δ^2 is a polynomial in $\mathbb{C}[e_1, \ldots, e_n]$ and find this polynomial explicitly when n is 2 or 3. Show that if $n = 3$ the ring of invariants is not a polynomial ring.

Example 316 Suppose G is the dihedral group of order $2n$ acting on R^2. We will find its ring of invariants. The obvious coordinate are x and y, but it is easier to use $z = x + iy$ and $\overline{z} = x - iy$ as coordinates. Then one generator of G takes z to ζz and $\overline{z}$ to $\zeta\overline{z}$, while the other exchanges z and $\overline{z}$. The ring of invariants is generated by $z\overline{z}$ and $z^n + \overline{z}^n$.

The invariants of a finite complex reflection group form a polynomial algebra. This was first proved by Shephard and Todd who classified all complex reflection groups into 3 infinite series and 34 exceptional cases, and found the ring of invariants cases by case. Shortly after Chevalley gave a uniform proof.

Lemma 317 *If H is a homogeneous polynomial in $I_1, \ldots$ and $\frac{\partial H}{\partial I_1}$ is a linear combination of $\frac{\partial H}{\partial I_2}, \ldots$, then $\frac{\partial H}{\partial I_1} = 0$.*

Proof Look at the terms of H containing a smallest power of I_1. This shows that H must contain terms not involving I_1. Moreover we get a linear relation between the terms of $\frac{\partial H}{\partial x_2} \ldots$ that do not involve I_1. Now use induction on $H(0, I_2, \cdots)$. □ Most rings

of invariants are not polynomial rings, so we need to find and use some special property of rings of invariants when the group is generated by reflections. The following is the special property of rings of invariants of groups generated by reflections that implies they are polynomial rings.

Lemma 318 *If $I_1, \dots$ are homogeneous invariants of a complex reflection group such that I_1 is not in the ideal generated by $I_2, \dots$, and there is some relations*

$$p_1 I_1 + p_2 I_2 + \cdots = 0$$

for homogeneous polynomials p_i, then p_1 is in the ideal generated by invariants of positive degree.

Proof The special property of reflections that we will use is that if g is a reflection with hyperplane $f = 0$ then $p - gp$ is divisible by f. So we find

$$\frac{gp_1 - p_1}{f} I_1 + \frac{gp_2 - p_2}{f} I_2 + \cdots = 0$$

and by induction on the degree of p_1 we see that $gp_1 - p_1$ is in the ideal generated by invariants of positive degree. So taking an average over $g \in G$ shows that $|G| p_1$ is a linear combination of an invariant and an element in the ideal generated by positive degree invariants. By assumption p_1 cannot have degree 0, so must be in the ideal generated by invariants of positive degree. □

Theorem 319 *(Chevalley-Shepherd-Todd) The ring of invariants of a complex reflection group is a polynomial algebra.*

Proof Pick a minimal set of homogeneous invariants $I_1, I_2, \dots$ that generate the ideal generated by positive degree invariants (so that by Hilbert's theorem they also generate the ring of invariants), We will show that they are algebraically independent. Suppose that $H(I_1, I_2, \dots) = 0$ for some homogeneous polynomial H. Then

$$\frac{\partial H}{\partial I_1} \frac{\partial I_1}{\partial x_i} + \frac{\partial H}{\partial I_2} \frac{\partial I_2}{\partial x_i} + \cdots = 0$$

Choose I_1 so that it is degree at least that of $I_2, \cdots$. Then $\frac{\partial H}{\partial I_1}$ is an invariant polynomial in $x_1, \dots$. It has degree at most equal to that of $\frac{\partial H}{\partial I_2}, \dots$ so by lemma ??? is is not in the ideal generated by $\frac{\partial H}{\partial I_2}, \dots$. By lemma ??? this shows that its coefficient $\frac{\partial I_1}{\partial x_i}$ is in the ideal generated by invariants of positive degree. But then

$$\deg(I_1) I_1 = \sum x_i \frac{\partial I_1}{\partial x_i}$$

is also in this ideal, contradicting the fact that I_1 is not in the ideal generated by the other invariants $I_2, \dots$. This shows that the invariants $I_1, I_2, \dots$ are algebraically independent, so the ring of invariants is the polynomial ring in these generators. □

Theorem 320 *(Molien) If G is a finite group acting on a vector space V, then the Poincar series of its ring of invariants $S(V)^G$ is*

$$\sum_n t^n \dim((S^n V)^G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - tg)}$$

Proof A single element g acting on $\oplus t^n S^n(V)$ has trace $\frac{1}{\det(1-tg)}$, as one can see by diagonalizing g. But the dimension of the invariant space of a representation is just the average of the trace of all elements of g. $\square$

In particular if the ring of invariants of a group is polynomial with generators of degrees d_i, we find

$$\prod \frac{1}{1-t^{d_i}} = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-tg)}$$

because the left hand side is the Poincar series of the polynomial ring. Multiplying both sides by $(1-t)^n$ for $n = \dim(V)$ we find

$$\prod \frac{1}{1+t+\cdots+t^{d_i}-1} = \frac{1}{|G|}\sum_{g\in G}\frac{(1-t)^n}{\det(1-tg)}$$

Setting $t = 1$ we find all terms on the left vanish except for the term corresponding to the identity element of G, so we find

$$\prod d_i = |G|.$$

Exercise 321 By examining the derivative of both sides at $t = 1$ show that

$$\sum(d_i - 1) = \text{half the number of (non-trivial) reflections of } G$$

The degrees d_i of the generating invariants of a Weyl group control the corresponding compact Lie group. For example, the real cohomology of the group is an exterior algebra on generators of degrees $2d_i - 1$, and the center of the UEA is a polynomial algebra generated by elements of degrees d_i and, the order of a "simply connected" Chevalley group is given by $q^{\sum d_i - 1}\prod(q^{d_i} - 1)$.

Example 322 The rings of invariants of groups that are not reflection groups tend to be rather complicated. For example, if we take a cyclic group of order n acting on $\mathbb{C}[x, y]$ where the generator multiplies both x and y by a primitive nth root of 1, then the invariants are generated by the $n+1$ monomials $x^i y^{n-i}$ which is usually not a polynomial ring. There are many relations between these generators. In general the ring of invariants of a finite group is finitely generated by Hilbert's theorem, but the number of generators can be very large, and is usually much larger than the dimension of the representation.

This still leaves the problem of finding the degrees of the polynomial generators of the Weyl groups of F_4, E_6, E_7, E_8. One way is to use the following fact: count the number of positive roots of given height (where the simple roots have height 1). Then the degrees of the invariants are the heights where the number of roots drops by 1.

Exercise 323 Find the heights of the 24 positive roots of F_4, and use the fact mentioned above to show that the degrees of the fundamnetal invariants are $2, 6, 8, 12$. If you are feeling ambitious, try E_6, E_7, E_8.