What is the order of the Weyl group of E_8? We'll do this by 4 different methods, which illustrate the different techniques for this kind of thing:

(1) This is a good one as a mnemonic. The order of E_8 is given by

$$\begin{aligned}|W(E_8)| &= 8! \times \prod \begin{pmatrix}\text{numbers on the}\\ \text{affine } E_8 \text{ diagram}\end{pmatrix} \times \frac{\text{Weight lattice of } E_8}{\text{Root lattice of } E_8}\\ &= 8! \times (1.2.3.4.5.6.4.2.3) \times 1\\ &= 2^{14} \times 3^5 \times 5^2 \times 7\end{aligned}$$

These are the numbers giving highest root.

We can do the same thing for any other Lie algebra, for example,

$$\begin{aligned}|W(F_4)| &= 4! \times (1.2.3.4.2) \times 1\\ &= 2^7 \times 3^2\end{aligned}$$

(2) The order of a reflection group is equal to the products of degrees of the fundamental invariants. For E_8, the fundamental invariants are of degrees 2,8,12,14,18,20,24,30 (primes +1).

(3) This one is actually an honest method (without quoting weird facts). The only fact we will use is the following: suppose G acts transitively on a set X with $H =$ the group fixing some point; then $|G| = |H| \cdot |X|$.

This is a general purpose method for working out the orders of groups. First, we need a set acted on by the Weyl group of E_8. Let's take the root vectors (vectors of norm 2). This set has 240 elements, and the Weyl group of E_8 acts transitively on it. So $|W(E_8)| = 240 \times |\text{subgroup fixing } (1, -1, 0^6)|$. But what is the order of this subgroup (call it G_1)? Let's find a set acted on by this group. It acts on the set of norm 2 vectors, but the action is not transitive. What are the orbits? G_1 fixes $s_1 = (1, -1, 0^6)$. For other roots r, G_1 obviously fixes (r, s). So how many roots are there with a given inner product with s?

(s,r)	number	choices
2	1	s
1	56	$(1, 0, \pm 1^6), (0, -1, \pm 1^6), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}^6)$
0	126	
-1	56	
-2	1	$-s$

So there are at least 5 orbits under G_1. In fact, each of these sets is a single orbit under G_1. How can we see this? Find a large subgroup of G_1. Take $W(D_6)$, which is generated by all permutations of the last 6 coordinates and all even sign changes of the last 6 coordinates. It is generated by reflections associated to the roots orthogonal to e_1 and e_2 (those that start with two 0s). The three cases with inner product 1 are three orbits under $W(D_6)$. To see that there is a single orbit under G_1, we just need some reflections that mess up these orbits. If we take a vector $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^6)$ and reflect norm 2 vectors through it, this mixes up the orbits under $W(D_6)$, so we get exactly 5 orbits. So G_1 acts transitively on these orbits.

In fact G_1 is the Weyl group of E_7, as we will see during the calculation. We also obtain the decomposition of the Lie algebra E_8 under the action of E_7: it splits as representations of dimensions 1, 56, 133, 1, 56, 1. If we look a bit more closely we see that in fact there is a subgroup $E7 \times SL_2$, and E_8 decomposes as $133{\otimes}1{\oplus}56{\otimes}2{\oplus}1{\otimes}3$. One can see directly from the roots that the 56 dimensional representation has an invariant bilinear form induced by the Lie bracket of E_8. The 56 dimensional representation of E_7 has the special property that all its weights are conjugate under the Weyl group: such representations are called minuscule, and tend to be rather special: they include spin representations and some vector representations.

We'll use the orbit of the 56 vectors r with $(r, s_1) = -1$. Let G_2 be the generated by reflections of vectors orthogonal to s_1 and s_2 where $S_2 = (0, 1, -1, 0, 0, 0, 0, 0)$.

We have that $|G_1| = |G_2| \cdot 56$.

G_2 is the Weyl group of E_6. We can see that E_7 decomposes under E_6 as $133 = 78 + 1 + 27 + 27$: we get two dual 27 dimensional minuscule representations of E_6. We can also decompose E_8 as a representation of E_6, or better as a representation of $E_6 + sl_2$, and we get $240 = 78 \times 1 + 27 \times 3 + 27 \times 3 + 1 \times 8$.

Our plan is to chose vectors acted on by G_i, fixed by G_{i+1} which give us the Dynkin diagram of E_8. So the next step is to try to find vectors r such that s_1, s_2, r form a Dynkin diagram A_3, in other words r has inner product -1 with s_2 and 0 with s_1. The possibilities for r are $(-1, -1, 0, 0^5)$ (one of these), $(0, 0, 1, \pm 1, 0^4)$ and permutations of its last five coordinates (10 of these), and $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^5)$ (there are 16 of these), so we get 27 total. Then we should check that they form one orbit, which is boring so we leave it as an exercise.

Next we find vectors r such that s_1, s_2, s_3, r form a Dynkin diagram A_4, where s_3 is of course $0, 0, 1, -1, 0, 0, 0, 0)$,

i.e., whose inner product is -1 with s_3 and zero with s_1, s_2. The possibilities are permutations of the last four coords of $(0, 0, 0, 1, \pm 1, 0^3)$ (8 of these) and $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^4)$ (8 of these), so there are 16 total. Again we should check transitivity, but ill not bother.

For the next step, we want vectors r such that s_1, s_2, s_3, s_4, r form a Dynkin diagram A_5; the possibilities are $(0^4, 1, \pm 1, 0^2)$ and permutations of the last three coords (6 of these), and $(-\frac{1}{2}^4, \frac{1}{2}, \pm\frac{1}{2}^3)$ (4 of these) for a total of 10 vectors r, and as usual these form a single orbit under G_5.

For the next step, we want vectors r such that $s_1, s_2, s_3, s_4, s_5, r$ form a Dynkin diagram A_6; the possibilities are $(0^5, 1, \pm 1, 0)$ and permutations of the last two coords (4 of these), and $(-\frac{1}{2}^5, \frac{1}{2}, \pm\frac{1}{2}^2)$ (2 of these) for a total of 6 vectors r, and as usual these form a single orbit under G_6.

The next case is tricky: we want vectors r such that $s_1, s_2, s_3, s_4, s_5, s_6, r$ form a Dynkin diagram A_7;, the possibilities are $(0^6, 1, \pm 1)$ (2 of these) and $((-\frac{1}{2})^6, \frac{1}{2}, \frac{1}{2})$ (just 1). The proof of transitivity fails at this point.

The group G_7 we are using by now doesn't even act transitively on the pair $(0^6, 1, \pm 1)$ (we can't get between them by changing an even number of signs). What elements of $W(E_8)$ fix all of these first 6 points? We want to find roots perpendicular to all of these vectors, and the only possibility is $((\frac{1}{2})^8)$. How does reflection in this root act on the three vectors above? $(0^6, 1^2) \mapsto ((-\frac{1}{2})^6, \frac{1}{2}^2)$ and $(0^6, 1, -1)$ maps to itself. Is this last vector in the same orbit? In fact they are in different orbits. To see this, look for vectors completing the E_8 diagram. In the $(0^6, 1, 1)$ case, we can take the vector $((-\frac{1}{2})^5, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. But in the other case, we can show that there are no possibilities. So these really are different orbits. In other words, there are 3 possible roots r, but these form two orbits under G_7 of sizes 1 and 2.

We use the orbit with 2 elements, and check that there are no automorphisms fixing s_1 to s_7, so we find

$$|W(E_8)| = 240 \times \underbrace{56 \times \overbrace{27 \times 16 \times 10 \times 6 \times 2 \times 1}^{\text{order of } W(E_6)}}_{\text{order of } W(E_7)}$$

because the group fixing all 8 vectors must be trivial. We also get that

$$|W(\text{“}E_5\text{”})| = 16 \times \underbrace{10 \times \overbrace{6 \times 2 \times 1}^{|W(A_2 \times A_1)|}}_{|W(A_4)|}$$

where "E_5" is the algebra with diagram (that is, D_5). Similarly, E_4 is A_4 and E_3 is $A_2 \times A_1$.

We got some other information. We found that the Weyl group of E_8 acts transitively on all the configurations A_1, A_2, A_3, A_4, A_5, A_6, but not on A_7. Obviously a similar method can be used to find orbits of other reflection groups on other configurations of roots.

The sequence of numbers 1, 2 (or 3), 6, 10, 16, 27, 56, 240 tends to turn up in a few other places, such as the number of exceptional curves on a del Pezzo surface (blow up the plane at some points). In particular the number 27 is the same 27 that appears in the 27 lines on a cubic surface (=plane blown up at 6 points).

(4)

We now give the fourth method of finding the order of $W(E_8)$. Let L be the E_8 lattice. Look at $L/2L$, which has 256 elements. Look at this as a set acted on by $W(E_8)$. There is an orbit of size 1 (represented by 0). There is an orbit of size $240/2 = 120$, which are the roots (a root is congruent mod $2L$ to it's negative). Left over are 135 elements. Let's look at norm 4 vectors. Each norm 4 vector, r, satisfies $r \equiv -r \mod 2$, and there are $240 \cdot 9$ of them, which is a lot, so norm 4 vectors must be congruent mod 2 to other norm 4 vectors. Let's look at $r = (2,0,0,0,0,0,0,0)$. Notice that it is congruent to vectors of the form $(0 \cdots \pm 2 \ldots 0)$, of which there are 16. It is easy to check that these are the only norm 4 vectors congruent to r mod 2. So we can partition the norm 4 vectors into $240 \cdot 9/16 = 135$ subsets of 16 elements. So $L/2L$ has 1+120+135 elements, where 1 is the zero, 120 is represented by 2 elements of norm 2, and 135 is represented by 16 elements of norm 4. A set of 16 elements of norm 4 which are all congruent is called a FRAME. It consists of elements $\pm e_1, \ldots, \pm e_8$, where $e_i^2 = 4$ and $(e_i, e_j) = 1$ for $i \neq j$, so up to sign it is an orthogonal basis.

Then we have

$$|W(E_8)| = (\#\text{ frames}) \times |\text{subgroup fixing a frame}|$$

because we know that $W(E_8)$ acts transitively on frames. So we need to know what the automorphisms of an orthogonal base are. A frame is 8 subsets of the form $(r, -r)$, and isometries of a frame form the group $(\mathbb{Z}/2\mathbb{Z})^8 \cdot S_8$, but these are not all in the Weyl group. In the Weyl group, we found a $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$, where the first part is the group of sign changes of an even number of coordinates. So the subgroup fixing a frame must be in between these two groups, and since these groups differ by a factor of 2, it must be one of them. Observe that changing an odd number of signs doesn't preserve the E_8 lattice, so it must be the group $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$, which has order $2^7 \cdot 8!$. So the order of the Weyl group is

$$135 \cdot 2^7 \cdot 8! = |2^7 \cdot S_8| \times \frac{\#\text{ norm 4 elements}}{2 \times \dim L}$$

Remark 312 Conway used a similar method to calculate the order of his largest simple group. In this case if we take the Leech lattice mod 2, it decomposes rather like E_8 mod 2 except there are 4 orbits: the zero vector, orbits represented by a pair $\pm r$ of norm 4 vectors, orbits represented by a pair $\pm r$ of norm 6 vectors, and orbits represented by a frame of 48 norm 8 vectors. The subgroup fixing a frame is $2^{12}.M_{24}$, If Λ is the Leech lattice, we find the order of its automorphism group is

$$|2^{12} \cdot M_{24}| \cdot \frac{\#\text{ norm 8 elements}}{2 \times \dim \Lambda}$$

where M_{24} is the Mathieu group (one of the sporadic simple groups). Conway's simple group has half this order, as one gets it by quotienting out the center ± 1. The Leech lattice seems very much to be trying to be the root lattice of the monster group, or something like that, with its automorphism group behaving rather like a Weyl group, but no one has really been able to make sense of this idea.

$W(E_8)$ acts on $(\mathbb{Z}/2\mathbb{Z})^8$, which is a vector space over $\mathbb{F}_2$, with quadratic form $N(a) = \frac{(a,a)}{2} \mod 2$, so we get a map

$$\pm 1 \to W(E_8) \to O_8^+(\mathbb{F}_2)$$

which has kernel ± 1 and is surjective, as can be seen by comparing the orders of both sides. O_8^+ is one of the 8 dimensional orthogonal groups over $\mathbb{F}_2$. So the Weyl group of E_8 is a double cover of an orthogonal group of a vector space over $\mathbb{F}_2$.