The Lie algebras we get like this are usually infinite dimensional, as they contain free lie algebras on E. The key idea is that if we have a subspace X of $Free(E)$ that is mapped to itself by F and H, then the subspace it generates under the action of E is still acted on by F and H, so is an ideal of G contained in $Free(E)$, and therefore having zero intersection with H. So if we can find such subspaces X we can reduce the size of G. In general there need not be any such subspaces, but for the special case of the Serre relations we can find some.

The final step is to include the relations $\mathrm{ad}(e_i)^{1-a_{ij}}e_j = 0$ $\mathrm{ad}(f_i)^{1-a_{ij}}f_j = 0$ and in particular we need to show that these do not cause the Lie algebra to collapse to 0.

Lemma 304 *If $i \neq j$ then the element $ad(e_i)^{1-a_{ij}}e_j = 0$ of E is is killed by f_k.*

Proof It is obviously killed by f_k if k is not i or j, as then f_k commutes with both e_i and e_j.

To show that it is killed by f_j we have two cases depending on whether a_{ij} is 0 or not.

$$
\begin{aligned}
[f_j[e_i, [\cdots[e_i, e_j]\cdots] &= [e_i, [\cdots[e_i, [f_j, e_j]\cdots] && (23)\\
&= -[e_i, [\cdots[e_i, h_j]\cdots] = a_{ij}[e_i, \ldots, , e_i]\cdots] && (24)
\end{aligned}
$$

If $a_{ij} = 0$ this vanishes as it contains a factor of a_{ij}, while if $a_{ij} > 0$ it vanishes because $1 - a_{ij} \geq 2$ so it contains a term $[e_i, e_i]$.

Finally we show that $\mathrm{ad}(e_i)^{1-a_{ij}}e_j = 0$ is killed by f_i, which is where we need to use the funny-looking exponent $1 - a_{ij}$. For this we look at the subalgebra generated by e_i, f_i, and h_i, which is isomorphic to $\mathfrak{sl}_2$. Moreover the element e_j is killed by f_i so generates a Verma module for $\mathfrak{sl}_2$. The lowest weight e_j of this Verma module has eigenvalue a_{ij} for h_i, so by the theory of $\mathfrak{sl}_2$ Verma modules, the element $\mathrm{ad}(e_i)^n e_j = 0$ for $n > 0$ is killed by f_j if (and only if) $n = 1 - a_{ij}$. □

Theorem 305 *In the Lie algebra defined by the Serre relations, the elements h_i, e_i, and f_i are linearly independent.*

Corollary 306 *Each of the copies of sl_2 spanned by e_i, f_i, h_i act on the Lie algebra as a sum of finite dimensional representations.*

Proof This follows by first checking that each generator of the Lie algebra lies in a finite dimensional representation (using the extra Serre relations) then showing that the elements of the Lie algebra with this property are closed under the Lie bracket. □

Corollary 307 *The action of each Lie algebra sl_2 spanned by e_i, f_i, h_i lifts to an action of the Lie group SL_2. In particular the Weyl group element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ acts on the Lie algebra, and acts on the Cartan subalgebra as the reflection of the corresponding simple root, so we get an action of the Weyl group on the roots.*

This is enough to show that the Lie algebra is finite dimensional: more precisely every root is conjugate to a simple root under the Weyl group and therefore has multiplicity 1. (This fails for infinite root systems: for general Kac-Moody algebras there are more roots that are not conjugate to simple roots, called imaginary roots, and they can have multiplicity greater than 1.)

Example 308 We can work this out explicitly for the rank 2 algebras, and write down explicit bases.

Theorem 309 *Every root is conjugate to a simple root under the Weyl group.*

Proof We need to use the fact that to root system and Weyl group are finite: the theorem fails for infinite dimensional Kac-Moody algebras, which have "imaginary" roots not conjugate to simple roots. In particular there is a positive definite quadratic form preserved by the Weyl group. Look at the Weyl chamber W of the simple roots, and its dual convex cone C generated by the simple roots. Any positive root is contained in C, and its orthogonal complement has codimension at least 2 in W unless the root is simple. The conjugates of the Weyl chamber W under the Weyl group cover the whole space, so if a root is not conjugate to a simple root under the Weyl group, its orthogonal complement has codimension at least 2 in the whole space , which is impossible. $\square$

To summarize, we have an explicit description of the Lie algebra: it is a sum of the Cartan subalgebra H (with basis h_i), and a 1-dimensional root space for each root of the root system, for which we can easily write down an explicit basis element if we want to.

Exercise 310 Show that the Lie algebra constructed from an irreducible finite crystallographic root system is simple. (Irreducible means that it is not the sum of two orthogonal root systems: in the case the Lie algebra splits as the direct sum of corresponding Lie algebras.) The idea is to look at eigenvectors of the Cartan subalgebra. If α is some eigenvalue of some element in an ideal, then show that so is β for and β not orthogonal to α.

Much, but not all, of this theory works for infinite root systems, and the corresponding Lie algebras are called Kac-Moody algebras. We still get an action of the Weyl group on the roots, but as mentioned above roots need not be conjugate to simple roots: the proof above fails because the union of Weyl chambers need not cover space when the Weyl group is infinite.

25 The Weyl groups of exceptional groups

We use a vector notation in which powers represent repetitions: so $(1^8) = (1,1,1,1,1,1,1,1)$ and $(\pm\frac{1}{2}^2, 0^6) = (\pm\frac{1}{2}, \pm\frac{1}{2}, 0,0,0,0,0,0)$.

Recall that E_8 has the Dynkin diagram

where each vertex is a root r with $(r,r) = 2$; $(r,s) = 0$ when r and s are not joined, and $(r,s) = -1$ when r and s are joined. We choose an orthonormal basis $e_1, \ldots, e_8$, in which the roots are as given.

We want to figure out what the root lattice L of E_8 is (this is the lattice generated by the roots). If we take $\{e_i - e_{i+1}\} \cup (-1^5, 1^3)$ (all the A_7 vectors plus twice the strange vector), they generate the D_8 lattice $= \{(x_1, \ldots, x_8) | x_i \in \mathbb{Z}, \quad \sum x_i \text{ even}\}$. So the E_8 lattice consists of two cosets of this lattice, where the other coset is $\{(x_1, \ldots, x_8) | x_i \in \mathbb{Z} + \frac{1}{2}, \quad \sum x_i \text{ odd}\}$.

Alternative version: If we reflect this lattice through the hyperplane $e_1^\perp$, then we get the same thing except that $\sum x_i$ is always even. We will freely use both characterizations, depending on which is more convenient for the calculation at hand.

We should also work out the weight lattice, which is the vectors s such that $(r, r)/2$ divides (r, s) for all roots r. Notice that the weight lattice of E_8 is contained in the weight lattice of D_8, which is the union of four cosets of D_8: D_8, $D_8 + (1, 0^7)$, $D_8 + (\frac{1}{2}^8)$ and $D_8 + (-\frac{1}{2}, \frac{1}{2}^7)$. Which of these have integral inner product with the vector $(-\frac{1}{2}^5, \frac{1}{2}^3)$? They are the first and the last, so the weight lattice of E_8 is $D_8 \cup D_8 + (-\frac{1}{2}, \frac{1}{2}^7)$, which is equal to the root lattice of E_8.

In other words, the E_8 lattice L is unimodular (equal to its dual L'), where the dual is the lattice of vectors having integral inner product with all lattice vectors. This is also true of G_2 and F_4, but is not in general true of Lie algebra lattices.

The E_8 lattice is even, which means that the inner product of any vector with itself is always even.

Even unimodular lattices in $\mathbb{R}^n$ only exist if $8|n$ (this 8 is the same 8 that shows up in the periodicity of Clifford groups). The E_8 lattice is the only example in dimension equal to 8 (up to isomorphism, of course). There are two in dimension 16 (one of which is $L \oplus L$, the other is $D_{16} \cup$ some coset). There are 24 in dimension 24, which are the Niemeier lattices. In 32 dimensions, there are more than a billion!

The Weyl group of E_8 is generated by the reflections through $s^\perp$ where $s \in L$ and $(s, s) = 2$ (these are called roots). First, let's find all the roots: $(x_1, \ldots, x_8)$ such that $\sum x_i^2 = 2$ with $x_i \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ and $\sum x_i$ even. If $x_i \in \mathbb{Z}$, obviously the only solutions are permutations of $(\pm1, \pm1, 0^6)$, of which there are $\binom{8}{2} \times 2^2 = 112$ choices. In the $\mathbb{Z} + \frac{1}{2}$ case, we can choose the first 7 places to be $\pm\frac{1}{2}$, and the last coordinate is forced, so there are 2^7 choices. Thus, we get 240 roots.

Let's find the orbits of the roots under the action of the Weyl group. We don't yet know what the Weyl group looks like, but we can find a large subgroup that is easy to work with. Let's use the Weyl group of D_8, which consists of the following: we can apply all permutations of the coordinates, or we can change the sign of an even number of coordinates (e.g., reflection in $(1, -1, 0^6)$ swaps the first two coordinates, and reflection in $(1, -1, 0^6)$ followed by reflection in $(1, 1, 0^6)$ changes the sign of the first two coordinates.)

Notice that under the Weyl group of D_8, the roots form two orbits: the set which is all permutations of $(\pm1^2, 0^6)$, and the set $(\pm\frac{1}{2}^8)$. Do these become the same orbit under the Weyl group of E_8? Yes; to show this, we just need one element of the Weyl group of E_8 taking some element of the first orbit to the second orbit. Take reflection in $(\frac{1}{2}^8)^\perp$ and apply it to $(1^2, 0^6)$: you get $(\frac{1}{2}^2, -\frac{1}{2}^6)$, which is in the second orbit. So there is just one orbit of roots under the Weyl group.

What do orbits of $W(E_8)$ on other vectors look like? We're interested in this because we might want to do representation theory. The character of a representation is a map from weights to integers, which is $W(E_8)$-invariant. Let's look at vectors of norm 4 for example. So $\sum x_i^2 = 4$, $\sum x_i$ even, and $x_i \in \mathbb{Z}$ or $x_i \in \mathbb{Z} + \frac{1}{2}$. There are 8×2 possibilities which are permutations of $(\pm 2, 0^7)$. There are $\binom{8}{4} \times 2^4$ permutations of $(\pm 1^4, 0^4)$, and there are 8×2^7 permutations of $(\pm\frac{3}{2}, \pm\frac{1}{2}^7)$. So there are a total of 240×9 of these vectors. There are 3 orbits under $W(D_8)$, and as before, they are all one orbit under the action of $W(E_8)$. Just reflect $(2,\, 0^7)$ and $(1^3, -1, 0^4)$ through $(\frac{1}{2}^8)$.

Exercise 311 Show that the number of norm 6 vectors is 240×28, and they form one orbit

(If you've seen a course on modular forms, you'll know that the number of vectors of norm $2n$ is given by $240 \times \sum_{d|n} d^3$. If we let call these c_n, then $\sum c_n q^n$ is a modular form of level 1 (E_8 even, unimodular), weight 4 ($\dim E_8/2$).)

For norm 8 there are two orbits, because we have vectors that are twice a norm 2 vector, and vectors that are not. As the norm gets bigger, there are a large number of orbits.