Which of these can be made into unitary representations? $H^\dagger = -H$, $E^\dagger = F$, and $F^\dagger = E$. If we have a Hermitian inner product $(\ ,\ )$, we see that

$$\begin{aligned}
(v_{j+2}, v_{j+2}) &= \frac{2}{\lambda+j+1}(Ev_j, v_{j+2}) \\
&= \frac{2}{\lambda+j+1}(v_j, -Fv_{j+2}) \\
&= -\frac{2}{\lambda+j+1}\overline{\frac{\lambda-j-1}{2}}(v_j, v_j) > 0
\end{aligned}$$

So we want $-\frac{\overline{\lambda-1-j}}{\lambda+j+1}$ to be real and positive whenever $j, j+2$ are non-zero eigenvectors. So

$$-(\lambda-1-j)(\lambda+1+j) = -\lambda^2 + (j+1)^2$$

should be positive for all j. Conversely, when we have this condition, the representations have a positive semi definite Hermitian form.

This condition is satisfied in the following cases:

1. $\lambda^2 \leq 0$. These representations are called PRINCIPAL SERIES representations. These are all irreducible *except* when $\lambda = 0$ and n is odd, in which case it is the sum of two limits of discrete series representations
2. $0 < \lambda < 1$ and j even. These are called COMPLEMENTARY SERIES. They are annoying, and you spend a lot of time trying to show that they don't occur in cases you are interested in, such as the Selberg conjecture.
3. $\lambda^2 = n^2$ for $n \geq 1$ (for some of the irreducible pieces).

   If $\lambda = 1$, we get a 1-dimensional subrepresentation, which is unitary, and the quotient is the sum of two Verma modules (discrete series representations).

   We see that we get two discrete series and a 1 dimensional representation, all of which are unitary

   For $\lambda = 2$ (this is the more generic one), we have a 2-dimensional middle representation (where $(j+1)^2 < \lambda^2 = 4$ that is not unitary, which we already knew. So the discrete series representations are unitary, and the finite dimensional representations of dimension greater than or equal to 2 are not.

Summary: the irreducible unitary representations of $SL_2(\mathbb{R})$ are given by

1. the 1 dimensional representation
2. Discrete series representations for any $\lambda \in \mathbb{Z} \setminus \{0\}$
3. Two limit of discrete series representations for $\lambda = 0$
4. Two series of principal series representations:

$$\begin{array}{l} j \text{ even: } \lambda \in i\mathbb{R},\ \lambda \geq 0 \\ j \text{ odd: } \lambda \in i\mathbb{R},\ \lambda > 0 \end{array}$$

5. Complementary series: parametrized by λ, with $0 < \lambda < 1$.

There seems to be a puzzle here: the discrete series representations of the group are (completions of) Verma modules, whereas we claimed earlier that Verma modules were never associated to representations of the group. The difference is that we are looking at Verma modules for different Cartan subalgebras. For the split Cartan subalgebra there is a Weyl group element acting as -1 on the Lie algebra, which implies that representations of the group cannot be Verma modules whose weights are not invariant under -1. On the other hand, for the compact Cartan subalgebra there is no element of the group acting as -1 on its Lie algebra, so this argument no longer applies, and we can have Verma modules that are (essentially) representations of the group, at least if we take a completion of them.

The unitary representations have an obvious topology (for general groups there is also such a topology, called the Fell topology). This topology is not Hausdorff: for example, the two limits of discrete series representations are limits of the same continuous series representations, and the two smallest discrete series representations and the trivial representation are all some sort of limit of complementary series representations as λ tens to 1.

The nice stuff that happened for $SL_2(\mathbb{R})$ breaks down for more complicated Lie groups. In particular if the rank is grater than 1 then the Casimir eigenvalue does not unambiguously determine what the analogues of the operators FE and so on do.

Representations of finite covers of $SL_2(\mathbb{R})$ are similar, except j need not be integral. For example, for the double cover $\widehat{SL_2(\mathbb{R})} = Mp_2(\mathbb{R})$, $2j \in \mathbb{Z}$.

Exercise 296 Find the irreducible unitary representations of $Mp_2(\mathbb{R})$.

Random things not covered: Plancherel measure for $SL_2(\mathbb{R})$ (note that 1 is not in the support!), holomorphic modular forms are highest weights for discrete series representations, Maass wave forms are eigenvectors for principal series representations, characters of representations.

24 Serre relations

We now construct all the simple complex Lie algebras using the Serre relations. The Cartan matrix of a root system with simple roots r_i is given by $a_{ij} = 2(r_i, r_j)/(r_i, r_i)$; these numbers are the integers that appear when reflecting r_j in the hyperplane of r_i.

Suppose that a_{ij} is a Cartan matrix. We can recover the Lie algebra with this matrix as the Lie algebra generated by elements h_i, e_i, f_i subject to the following Serre relations:

$$[e_i, f_j] = h_i \text{if } i = j, 0 \text{otherwise} \tag{15}$$

$$[h_i, e_j] = a_{ij} e_j \tag{16}$$

$$[h_i, f_j] = -a_{ij} e_j \tag{17}$$

$$\mathrm{ad}(e_i)^{1-a_{ij}} e_j = 0 \tag{18}$$

$$\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0 \tag{19}$$

As usual when objects are defined using a presentation it is easy to find an upper bound on the size of the object but harder to find a lower bound on the size. In particular the main problem with the algebra defined by the Serre relations is to show that it does not collapse to zero. We will do this in two steps: first show that in the algebra generated by the first 3 relations the elements h_i are linearly independent by finding some explicit representations of it, then showing that the ideal generated by the last two relations has trivial intersection with the subalgebra H spanned by the h_i.

So we first forget about the last two relations. Suppose that we have a graded lie algebra, and let F, H and E be the pieces of degree $-1, 0, 1$. Then

- H is a Lie algebra
- E and F are representations of H
- we have a map $E \otimes F \mapsto H$ of H-modules.

Typical example: H=diagonal matrices, E=the things just above the diagonal, and F=things just below. In applications, H will be the (abelian) Cartan subalgebra, E with be the sum of the simple root spaces, and F with be the sum of the root spaces of minus the simple roots.

Conversely given the data above, we want to construct a graded Lie algebra G. Obviously we can define a universal such G by generators and relations, and the problem is to see what its structure is: in particular is is not obvious that G is nonzero!

We first bound G from above, which is straightforward.

Lemma 297 *The obvious vector space map from $Free(F) \oplus H \oplus Free(E)$ to G is onto, where Free means "free Lie algebra generated by a vector space".*

Proof This is easy: just check that the image of this map is closed under brackets by elements of E, F, and H, by induction on length. □

Now we have to show that this map is injective, and in particular that G does not collapse to nothing. As usual, the best way to show that something given by generators and relations is non-zero is to construct a representation of it. The idea is to construct representations as some sort of Verma modules: in other words we have a "lowest weight space" V acted on by H and killed by F, where V is any representation of H. This gives the action of F and H, but we have no idea what E does. We try to build a representation by letting the action of E be as free as possible: in other words we take $TE \otimes V$ where TE is the tensor algebra of E (the UEA of the free Lie algebra generated by E). First we need a lemma for constructing operators on TE:

Lemma 298 *Suppose that E is the free associative algebra generated by elements e_i, and we are given operators b_i on E and an element e of E. Then there is a unique operator b such that $b(1) = e$ and $[b, e_i] = b_i$.*

Proof The algebra E has a basis of elements $e_{i_1} e_{i_2} \cdots e_{i_n}$ for $n \geq 0$. We define b by induction on the length n of a basis element by putting

$$b(1) = e$$

for the basis element of length 0, and

$$b(e_i \cdots) = e_i b(\cdots) + b_i(\cdots)$$

on elements of positive length. □

So if we know that action of some operator on V, and we are given its commutators with elements of E (which should of course depend linearly on E), then we get a unique operator on $TE \otimes V$. This immediately gives us operators on $TE \otimes V$ corresponding to elements of H and F, using the fact that we know $[h, e]$ and then $[f, e]$. To finish the proof that we have a representation, we need to check that $[h_1, h_2]$ and $[h, f]$ have the right values. The idea for doing this is that if two operators on $TE \otimes V$ are the same on V and have the same commutators with elements of E, then they are equal. If we write x' for the operator on $TE \otimes V$ corresponding to x, this means we have to check that $[h'_1, h'_2] = [h_1, h_2]'$ and $[h, f]' = [h', f']$. For both identities it is immediate that they coincide on V, so we just have to check the commutators with and $e \in E$ are the same. This follows from

$$[e', [h'_1, h'_2]] = [[e', h'_1], h'_2] + [h'_1, [e', h'_2]] = [[e, h_1]', h'_2] + [h'_1, [e, h_2]'] = [[e, h_1], h_2]' + [h_1, [e, h_2]]' = [e, [h_1, h_2]]' =$$

since we know that $'$ is a homomorphism brackets of type $[E, H]$.

Exercise 299 Do the case of $[h, f]' = [h', f']$ in the same way.

Exercise 300 Where did the proof use the fact that the map from $E \otimes F \mapsto h$ is a homomorphism of H-modules?

In particular we see that $Free(E)$ acts freely on these modules so maps injectively into G, and by taking V to be any faithful representation of H we see that H also maps injectively into G. To summarize, we have proved:

Theorem 301 *Suppose given a Lie algebra H, and H-modules E and F together with a map of H-modules from $E \otimes F$ to H. Then E, F, and H are the pieces of degree 1, -1 and 0 of a graded Lie algebra G, and the positive part of G is the free Lie algebra on E. Moreover if V is any representation of H, we can extend it to a representation of G on $TE \otimes V$ so that F kills V and E acts in the obvious way by left multiplication.*

Lemma 302 *Suppose that $\mathfrak{g}$ is the Lie algebra generated by elements generated by elements h_i, e_i, f_i subject to the following relations:*

$$[e_i, f_j] = h_i if\ i = j, 0 otherwise \tag{20}$$
$$[h_i, e_j] = a_{ij} e_j \tag{21}$$
$$[h_i, f_j] = -a_{ij} e_j \tag{22}$$

Then $\mathfrak{g}$ can be graded as $\mathfrak{g} = \oplus \mathfrak{g}_m$, where $\mathfrak{g}_+ = \oplus_{m>0} \mathfrak{g}_m$ is the subalgebra generated by the e_i, $\mathfrak{g}_- = \oplus_{m>0} \mathfrak{g}_m$ is the subalgebra generated by the f_i, and $\mathfrak{g}_0$ is the subalgebra generated by the h_i and is abelian.

Proof This is a fairly straightforward check.

We define the grading by giving each e_i degree 1, each f_i degree -1, and each h_i degree 0. This defines a grading as all the relations are homogeneous. (More generally, we could give e_i and positive degree d_i provided we give f_i degree $-d_i$, which is occasionally useful.)

The fact that the subalgebra H generated by the elements h_i is abelian follows from

$$[h_i, h_j] = [h_i, [e_j, f_j]] = [[h_i, e_j], f_j] + [e_j, [h_i, f_j]] = [a_{ij}e_j, f_j] + [e_j, [-a_{ij}f_j] = 0$$

so all the elements h_i commute with each other, and in particular H is spanned by the elements h_i.

If we write E, F, and $H = g_0$ for the subalgebras generated by the elements e_i, f_i, and h_i then the defining relations imply $[f_i, E] \subseteq E$, $[f_i, E] \subseteq H \oplus E$, $[f_i, H] \subseteq F$, so the subspace $E \oplus H \oplus F$ is closed under f_i. Similarly it is closed under e_i and is therefore equal to $\mathfrak{g}$ as these elements generate $\mathfrak{g}$. □

Now we come to the first key point, which is showing that the algebra $\mathfrak{g}$ does not collapse.

Lemma 303 *The elements h_i of the Lie algebra $\mathfrak{g}$ are linearly independent.*

Proof This follows from the previous theorem. □