23 Infinite dimensional unitary representations

Last lecture, we found the finite dimensional (non-unitary) representations of $SL_2(\mathbb{R})$.

23.1 Background about infinite dimensional representations

(of a Lie group G) What is an infinite dimensional representation?

1st guess Banach space acted on by G?

This is no good for the following reasons: Look at the action of G on the functions on G (by left translation). We could use L^2 functions, or L^1 or L^p. These are completely different Banach spaces, but they are essentially the same representation.

2nd guess Hilbert space acted on by G? This is sort of okay.

The problem is that finite dimensional representations of $SL_2(\mathbb{R})$ are NOT Hilbert space representations, so we are throwing away some interesting representations.

Solution (Harish-Chandra) Take $\mathfrak{g}$ to be the Lie algebra of G, and let K be the maximal compact subgroup. If V is an infinite dimensional representation of G, there is no reason why $\mathfrak{g}$ should act on V.

The simplest example fails. Let $\mathbb{R}$ act on $L^2(\mathbb{R})$ by left translation. Then the Lie algebra is generated by $\frac{d}{dx}$ (or $i\frac{d}{dx}$) acting on $L^2(\mathbb{R})$, but $\frac{d}{dx}$ of an L^2 function is not in L^2 in general.

Let V be a Hilbert space. Set V_ω to be the K-finite vectors of V, which are the vectors contained in a finite dimensional representation of K. The point is that K is compact, so V splits into a Hilbert space direct sum finite dimensional representations of K, at least if V is a Hilbert space. Then V_ω is a representation of the Lie algebra $\mathfrak{g}$, not a representation of G. V_ω is a representation of the group K. It is a $(\mathfrak{g}, K)$-module, which means that it is acted on by $\mathfrak{g}$ and K in a "compatible" way, where compatible means that

1. they give the same representations of the Lie algebra of K.
2. $k(u)v = k(u(k^{-1}v))$ for $k \in K$, $u \in \mathfrak{g}$, and $v \in V$.

The K-finite vectors of an irreducible unitary representation of G is ADMISSIBLE, which means that every representation of K only occurs a *finite* number of times. The GOOD category of representations is the representations of admissible $(\mathfrak{g}, K)$-modules. It turns out that this is a really well behaved category.

We want to find the unitary irreducible representations of G. We will do this in several steps:

1. Classify all irreducible admissible representations of G. This was solved by Langlands, Harish-Chandra et. al.

2. Find which have Hermitian inner products (,). This is easy.
3. Find which ones are positive definite. This is very hard, and has not been solved for all simple Lie groups, though it has been done for some infinite series such as general linear groups. We will only do this for the simplest case: $SL_2(\mathbb{R})$, which is much easier than most other cases.

23.2 The group $SL_2(\mathbb{R})$

We found some generators (in $Lie(SL_2(\mathbb{R}))\otimes\mathbb{C}$ last time: E, F, H, with $[H,E] = 2E$, $[H,F] = -2F$, and $[E,F] = H$. We have that $H = -i\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $E = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$, and $F = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$. Why not use the old $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$?

Because $SL_2(\mathbb{R})$ has two different classes of Cartan subgroup: $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and the second one is COMPACT. The point is that non-compact (abelian) groups need not have eigenvectors on infinite dimensional spaces. An eigenvector is the same as a weight space. The first thing you do is split it into weight spaces, and if your Cartan subgroup is not compact, you cannot get started. We work with the compact subalgebra so that the weight spaces exist.

Given the representation V, we can write it as some direct sum of eigenspaces of H, as the Lie group H generates is compact (isomorphic to S^1). In the finite dimensional case, we found a HIGHEST weight, which gave us complete control over the representation. The trouble is that in infinite dimensions, there is no reason for the highest weight to exist, and in general they do not as there may be an infinite number of eigenvalues.

A good substituted for the highest weight vector: Look at the Casimir operator $\Omega = 2EF + 2FE + H^2 + 1$. The key point is that Ω is in the center of the universal enveloping algebra. As V is assumed admissible, we can conclude that Ω has eigenvectors (because we can find a finite dimensional space acted on by Ω). As V is irreducible and Ω commutes with G, all of V is an eigenspace of Ω. We will see that this gives us about as much information as a highest weight vector.

Let the eigenvalue of Ω on V be λ^2 (the square will make the most interesting representations have integral λ; the $+1$ in Ω is for the same reason).

Suppose $v \in V_n$, where V_n is the space of vectors where H has eigenvalue n. In the finite dimensional case, we looked at Ev, and saw that $HEv = (n+2)Ev$. What is FEv? If v was a highest weight vector, we could control this. Notice that $\Omega = 4FE + H^2 + 2H + 1$ (using $[E,F] = H$), and $\Omega v = \lambda^2 v$. This says that $4FEv + n^2 v + 2nv + v = \lambda^2 v$. This shows that FEv is a multiple of v.

Now we can draw a picture of what the representation looks like: There is a basis $\ldots v_{n-2}, v_n, v_{n+2}, \ldots$, with

- $Hv_n = nv_n$
- $Ev_n =$ some multiple of v_{n+2}
- $Fv_n =$ some multiple of v_{n-2}

Thus, V_ω is spanned by V_{n+2k}, where k is an integer. The non-zero elements among the V_{n+2k} are linearly independent as they have different eigenvalues. The only question remaining is whether any of the V_{n+2k} vanish.

There are four possible shapes for an irreducible representation

- infinite in both directions:
- a lowest weight, and infinite in the other direction:
- a highest weight, and infinite in the other direction:
- we have a highest weight and a lowest weight, in which case it is finite dimensional

We will see that all these show up. We also see that an irreducible representation is completely determined once we know λ and some n for which $V_n \neq 0$. The remaining question is to construct representations with all possible values of $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}$. n is an integer because it must be a representations of the circle group.

We can write down explicit representations as follows, by copying the formulas for Verma modules and not cutting them off at the highest weight. We get a representatino with basis v_i where i runs through either all even integers or all odd integers, and the action is given by

- $Hv_n = nv_n$
- $Ev_n = \frac{\lambda+n+1}{2}v_{n+2}$
- $Fv_n = \frac{\lambda-n-1}{2}v_{n-2}$

It is easy to check that these maps satisfy $[E, F] = H$, $[H, E] = 2E$, and $[H, F] = -2F$

Problem: These may not be irreducible, and we want to decompose them into irreducible representations. The only way they can fail to be irreducible if if $Ev_n = 0$ of $Fv_n = 0$ for some n (otherwise, from any vector, we can generate the whole space). The only ways that can happen is if

$$\begin{array}{l} n \text{ even: } \lambda \text{ an odd integer} \\ n \text{ odd: } \lambda \text{ an even integer.} \end{array}$$

What happens in these cases? The easiest thing is probably just to write out an example.

Example 293 Take n even, and $\lambda = 3$, so we have two submodules: one with basis $v_4, v_6, \cdots$, and the other with basis $v_{-4}, v_{-6}, \ldots$. So V has two irreducible subrepresentations V_- and V_+, and $V/(V_- \oplus V_+)$ is an irreducible 3 dimensional representation with basis v_{-2}, v_0, v_2.

Example 294 If n is even, but λ is negative, say $\lambda = -3$, we get a subrepresentation with basis v_{-2}, v_0, v_2.

Here we have an irreducible finite dimensional representation. If we quotient out by that subrepresentation, we get $V_+ \oplus V_-$. So this is like the previous example, except that it has been turned upside down.

In particular we can see that the representations are not completely reducible in general.

There is one case when something slightly different happens:

Exercise 295 Show that for n odd, and $\lambda = 0$, then V splits are a direct sum of two irreducible submodules: $V = V_+ \oplus V_-$. (These are called limits of discrete series representations.)

So we have a complete list of all irreducible admissible representations:

1. if $\lambda \notin \mathbb{Z}$, we get one representation (remember $\lambda \equiv -\lambda$). This is the bi-infinite case.

2. Finite dimensional representation for each $n \geq 1$ ($\lambda = \pm n$)

3. Discrete series for each $\lambda \in \mathbb{Z} \setminus \{0\}$, which is the half infinite case: we get a lowest weight when $\lambda < 0$ and a highest weight when $\lambda > 0$.

4. two "limits of discrete series" where n is odd and $\lambda = 0$.