We will now find the irreducible finite dimensional representations of the Lie algebra $\mathfrak{sl}_2\mathbb{R}$, which has basis $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. H is a basis for the Cartan subalgebra $\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$. E spans the root space of the simple root. F spans the root space of the negative of the simple root. We find that $[H,E] = 2E$, $[H,F] = -2F$ (so E and F are eigenvectors of H), and you can check that $[E,F] = H$.

The Weyl group is generated by $\omega = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\omega^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Let V be a finite dimensional irreducible complex representation of $\mathfrak{sl}_2\mathbb{R}$. First decompose V into eigenspaces of the Cartan subalgebra (weight spaces) (i.e. eigenspaces of the element H). Note that eigenspaces of H exist because V is finite dimensional and complex. Look at the LARGEST eigenvalue of H (exists since V is finite dimensional), with eigenvector v. We have that $Hv = nv$ for some n. Compute

$$\begin{aligned} H(Ev) &= [H,E]v + E(Hv) \\ &= 2Ev + Env = (n+2)Ev \end{aligned}$$

So $Ev = 0$ (otherwise it would be an eigenvector of H with higher eigenvalue). $[E,-]$ increases weights by 2 and $[F,-]$ decreases weights by 2, and $[H,-]$ fixes weights.

We have that E kills v, and H multiplies it by n. What does F do to v?

What is $E(Fv)$?

$$\begin{aligned} EFv &= FEv + [E,F]v \\ &= 0 + Hv = nv \end{aligned}$$

In general, we have

$$\begin{aligned} H(F^i v) &= (n-2i)F^i v \\ E(F^i v) &= (ni - i(i-1))F^{i-1} v \\ F(F^i v) &= F^{i+1} v \end{aligned}$$

So the vectors $F^i v$ span V because they span an invariant subspace. This gives us an infinite number of vectors in distinct eigenspaces of H, and V is finite dimensional. Thus, $F^k v = 0$ for some k. Suppose k is the smallest integer such that $F^k v = 0$. Then

$$0 = E(F^k v) = (nk - k(k-1))\underbrace{EF^{k-1}v}_{\neq 0}$$

So $nk - k(k-1) = 0$, and $k \neq 0$, so $n - (k-1) = 0$, so $\boxed{k = n+1}$. So V has a basis consisting of $v, Fv, \dots, F^n v$. The formulas become a little better if we use the basis $w_n = v, w_{n-2} = Fv, w_{n-4} = \frac{F^2 v}{2!}, \frac{F^3 v}{3!}, \dots, \frac{F^n v}{n!}$.

This says that $E(w_2) = 5w_4$ for example. So we've found a complete description of all finite dimensional irreducible complex representations of $\mathfrak{sl}_2\mathbb{R}$.

These representations all lift to the group $SL_2(\mathbb{R})$: $SL_2(\mathbb{R})$ acts on homogeneous polynomials of degree n by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f(x,y) = f(ax+by, cx+dy)$. This is an $n+1$ dimensional space, and you can check that the eigenspaces are $x^i y^{n-i}$.

We have implicitly constructed Verma modules. We have a basis

$$w_n, w_{n-2}, \dots, w_{n-2i}, \dots$$

with relations

$$H(w_{n-2i}) = (n-2i)w_{n-2i}$$

,

$$Ew_{n-2i} = (n-i+1)w_{n-2i+2}$$

, and

$$Fw_{n-2i} = (i+1)w_{n-2i-2}$$

. These are obtained by copying the formulas from the finite dimensional case, but allow it to be infinite dimensional. This is the universal representation generated by the highest weight vector w_n with eigenvalue n under H (highest weight just means $E(w_n) = 0$).

For general semisimple groups, a Verma module is the universal module generated by a highest weight vector: a vector that is an eigenvector of the Cartan subalgebra (eigenvalue=weight) and killed by positive root spaces. They are easy to handle because one can start at the highest weight vector and work down. Any finite-dimensional irreducible module has a highest weight, so is a quotient of some Verma module. This suggests that it is useful to study the submodule structure of Verma modules. In general this is quite complicated: the solution involves Kazhdan-Lusztig polynomials, but for $SL_2(\mathbb{R})$ it is much easier and we can do it as follows.

We can start by asking when a Verma module V_λ with some highest weight λ is irreducible. So it has a basis v_λ, $v_{\lambda-2}, \ldots$, If we have some submodule W then by decomposing W into eigenspaces of H we may assume that W contains some eigenvector v_μ. Take μ to be as large as possible, so μ is killed by E. So we want to solve the following problem: which vectors v_μ are highest weight vectors? This is easy using the explicit formula $Ew_{\lambda-2i} = (\lambda - i + 1)w_{\lambda-2i+2}$ for the action of E: we see that it is only possible if $\lambda - i + 1 = 0$ (or $i = 0$) for some positive integer i. So generically Verma modules are irreducible: the only exceptions are the modules V_λ for λ a non-negative integer, which have a unique nonzero proper submodule isomorphic to $V_{-\lambda-2}$.

There is an alternative argument for testing when one Verma module is in another, which generalizes better to higher rank Lie algebras. For this we observe that the Casimir operator acts on a Verma module as multiplication by a scalar, so one Verma module maps non-trivially to another only if they have the same eigenvalue for the Casimir. To calculate this eigenvalue it is enough to do so for the highest weight vector, as this generates the Verma module and the Casimir commutes with the Lie algebra. To compute the eigenvalue on the highest weight vector it is convenient to rewrite the Casimir as

$$\Omega = 2EF + 2FE + H^2 = 2FE + 2[E,F] + 2FE + H^2 = 4FE + H^2 + 2H$$

because the term FE vanishes on the highest weight vector. So if the highest weight is λ, then the Casimir acts as multiplication by $\lambda^2 + 2\lambda = (\lambda+1)^2 - 1^2$. So if there is a non-zero map between two Verma modules with highest weights λ, μ then $(\lambda+1)^2 = (\mu+1)^2$, in other words $\lambda+1$ and $\mu+1$ are conjugate under the Weyl group $\{\pm 1\}$. For higher rank Lie algebras the analogous theorem says that two Verma modules with highest weights λ and μ have the same eigenvalues for ALL Casimir operators if and only if $\lambda + \rho$ and $\mu + \rho$ are conjugate under the Weyl group, where ρ is a vector called the Weyl vector, equal to half the sum of the positive roots.

The quotient $V_\lambda/V_{-\lambda-2}$ is the finite dimensional module of dimension $\lambda+1$. So we get an exact sequence

$$0 \to V_{-\lambda-2} \to V_\lambda \to (\lambda+1) - \dim \text{rep}$$

Verma modules are rank 1 free modules over the universal enveloping algebra of F, with character $q^\lambda/(1-q^2)$. If we pretend we do not know the character of the finite dimensional modules, we can work it out from this resolution by Verma modules by taking the alternating sum over the characters of the Verma modules, so we get $(q^\lambda - q^{-\lambda-2})/(1-q^2)$. The number of Verma modules appearing in this resolution is the order of the Weyl group. In fact if we look at the character formulas

$$\frac{\sum_w e^{w(\lambda+\rho)}}{\prod_{\alpha>0} e^\rho(1-e^{-\alpha})} = \sum_w \frac{e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})}$$

we see that it is expressing the character of a finite dimensional representation as an alternating sum of characters of Verma modules with highest weights $w(\lambda+\rho)-\rho$, coming from a resolution.

The factor $q-q^{-1}$ in the denominator of the character formula appears for completely different reasons in these two approaches to the representation theory. In the analytic approach where we integrate over compact groups, $q-q^{-1}$ appears as the square root of the fudge factor needed to convert integrals over the group to integrals over the Cartan subgroup. In the algebraic approach, it appears as the inverse of the character of a Verma module.

Let's look at some things that go wrong in infinite dimensions.

Warning 287 Representations corresponding to the Verma modules never lift to representations of $SL_2(\mathbb{R})$, or even to its universal cover. The reason: look at the Weyl group (generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$) of $SL_2(\mathbb{R})$ acting on $\langle H\rangle$; it changes H to $-H$. It maps eigenspaces with eigenvalue m to eigenvalue $-m$. But if you look at the Verma module, it has eigenspaces $n, n-2, n-4, \ldots$, and this set is obviously not invariant under changing sign. The usual proof that representations of the Lie algebra lifts uses the exponential map of matrices, which doesn't converge in infinite dimensions.

Remark 288 The universal cover $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$, or even the double cover $Mp_2(\mathbb{R})$, has no faithful finite dimensional representations. **Proof** Any finite dimensional representation comes from a finite dimensional representation of the Lie algebra $\mathfrak{sl}_2\mathbb{R}$. All such finite dimensional representations factor through $SL_2(\mathbb{R})$. $\square$

There is an obvious integral form of the UEA of SL_2, given as the one generated by E, F and H, in other words the UEA of the Lie algebra over the integers. However this is not really a very good integral form: for example, its dual (a commutative ring) ought to be something like a completion of a coordinate ring of the group, but is not. The right integral form was found by Chevalley, and can be motivated by trying to find an integral from that preserves the obvious integral form of the finite-dimensional representations.

Exercise 289 Show that the elements $E^n/n!$, $F^n/n!$ preserve the integral forms of finite dimensional representations and Verma modules given above. (The integral form of Verma modules is not the one generated by the action of F on the highest weight vector!)

So Chevalley's integral form of the universal enveloping algebra is the subalgebra generated by these elements. It has the useful property that coefficients of $\exp(tE)$ and $\exp(tF)$ are in the integral form. Since E and F act nilpotently on finite dimensional representations this means that $\exp(tE)$ and $\exp(tF)$ make sense over any commutative ring.

Exercise 290 Show that $H^n/n!$ does not usually preserve the integral form of finite dimensional representations, but that $\binom{H}{n}$ does. Show that these elements are in the integral form generated by $E^n/n!$ and $F^n/n!$.

Exercise 291 Show that the dual of the integral form above is a power series ring in 3 variables. (The integral form is a cocommutative Hopf algebra, so its dual is a commutative ring.)