22 Representations of SL2

Finite dimensional complex representations of the following are much the same: $SL_2(\mathbb{R})$, $\mathfrak{sl}_2\mathbb{R}$, $\mathfrak{sl}_2\mathbb{C}$ (as a complex Lie algebra), $\mathfrak{su}_2\mathbb{R}$, and SU_2. This is because finite dimensional representations of a simply connected Lie group are in bijection with representations of the Lie algebra. Complex representations of a REAL Lie algebra L correspond to complex representations of its complexification $L \otimes \mathbb{C}$ considered as a COMPLEX Lie algebra.

Note that complex representations of a COMPLEX Lie algebra $L \otimes \mathbb{C}$ are not the same as complex representations of the REAL Lie algebra $L \otimes \mathbb{C} \cong L + L$. The representations of the real Lie algebra correspond roughly to (reps of L)$\otimes$(reps of L).

Strictly speaking, $SL_2(\mathbb{R})$ is not simply connected, which is not important for finite dimensional representations.

Set $\Omega = 2EF + 2FE + H^2 \in U(\mathfrak{sl}_2\mathbb{R})$. The main point is that Ω commutes with $\mathfrak{sl}_2\mathbb{R}$. You can check this by brute force:

$$\begin{aligned}
[H, \Omega] &= 2\underbrace{([H,E]F + E[H,F])}_{0} + \cdots \\
[E, \Omega] &= 2[E,E]F + 2E[F,E] + 2[E,F]E \\
&\quad + 2F[E,E] + [E,H]H + H[E,H] = 0 \\
[F, \Omega] &= \text{Similar}
\end{aligned}$$

Thus, Ω is in the center of $U(\mathfrak{sl}_2\mathbb{R})$. In fact, it generates the center. This does not really explain where Ω comes from. Why does Ω exist? The answer is that it comes from a symmetric invariant bilinear form on the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ given by $(E,F) = 1$, $(E,E) = (F,F) = (F,H) = (E,H) = 0$, $(H,H) = 2$. This bilinear form is an invariant map $L \otimes L \to \mathbb{C}$, where $L = \mathfrak{sl}_2\mathbb{R}$, which by duality gives an invariant element in $L \otimes L$, which turns out to be $2E \otimes F + 2F \otimes E + H \otimes H$. The invariance of this element corresponds to Ω being in the center of $U(\mathfrak{sl}_2\mathbb{R})$.

The bilinear form on $SL_2(\mathbb{R})$ in turn can be constructed as $(a,b) = \mathrm{Trace}_V(ab)$ for some representation V. When V is the adjoint representation this is the Killing form. By a deep theorem of Cartan this form is non-degenerate when the Lie algebra is semisimple, though of course for $SL_2(\mathbb{R})$ this is easy to check directly.

Since Ω is in the center of $U(\mathfrak{sl}_2\mathbb{R})$, it acts on each irreducible representation as multiplication by a constant. We can work out what this constant is for the finite dimensional representations. Apply Ω to the highest vector w_n:

$$\begin{aligned}
(2EF + 2FE + HH)w_n &= (2n + 0 + n^2)w_n \\
&= (2n + n^2)w_n
\end{aligned}$$

So Ω has eigenvalue $2n+n^2$ on the irreducible representation of dimension $n+1$. Thus, Ω has DISTINCT eigenvalues on different irreducible representations, so it can be used to separate different irreducible representations. For more general semisimple Lie groups, the Casimir operator may take the same value on different irreducible representations, though it always distinguishes the trivial 1-dimensional representation from the others.

Theorem 282 *Finite dimensional representations of the complex Lie algebra* $sl_2(\mathbb{C})$ *are completely reducible.*

This is the key property that makes the representation theory easy. In particular the representation theory of this non-abelian Lie algebra is easier than that of apparently simpler algebras such as the abelian Lie algebra R^2 (classification of 2 commuting matrices is a hard problem).

Proof We will give two proofs of this result, both of which use important ideas.

For the first proof, we use the fact that all finite dimensional representations of compact groups are completely reducible. Since the finite dimensional complex representations of the complex Lie algebra $sl_2(C)$ are "the same" as the finite dimensional complex representations of the real Lie algebra of $SU(2)$, which are in turn the same as the finite-dimensional representations of the compact group $SU(2)$, its finite dimensional representations are completely reducible. (Its infinite dimensional representations are quite unlike those of $SU(2)$, and are not completely reducible.) This is Weyl's famous unitarian trick.

The second proof uses the Casimir operator, and illustrates how to use elements of the center of the UEA. This is an algebraic proof, that also works for some infinite dimensional Lie algebras when the "analytic" proof fails. The key point is that the Casimir operator can be used to separate the different irreducible representations, and in particular can separate the trivial representation from the others.

The key case is to show that if we have an exact sequence of modules

$$0 \to V \to W \to C \to 0$$

with V simple, then it splits. If V is the trivial 1-dimensional module, then this follows because $SL_2(C)$ is prefect: it has no nontrivial 2-dimensional representations that are strictly upper triangular. If V is nontrivial we use the Casimir operator: it has different eigenvalues for V and C, so W can be split as the sum of eigenspaces of the Casimir, and this splitting is invariant under $sl_2(C)$ because the Casimir commutes with $sl_2(C)$.

The general case follows from the key case above by linear algebra as follows. Any exact sequence of the form

$$0 \to V \to W \to C \to 0$$

for a possibly reducible V splits by induction on the length of V: we can split off a top irreducible component of V and work down. Now if we have a general exact sequence of the form

$$0 \to X \to Y \to Z \to 0$$

We want to find a splitting of this sequence, which is given by a $sl_2(C)$-invariant map from Y to X that is the identity on X. we let V be the subspace of $Hom_C(Y, X)$ of elements that act as a constant on $X \subseteq Y$, and let W be the codimension 1 subspace of elements where this constant is 0, so we have an exact sequence

$$0 \to V \to W \to C \to 0.$$

This splits, in other words we get a map from C to W whose image is fixed by $sl_2(C)$, in other words a $sl_2(C)$ linear map from Y to X that is a (nonzero!) constant on X. This gives the desired splitting of

$$0 \to X \to Y \to Z \to 0.$$

□

There are two properties of $sl_2(C)$ that make the proof of complete reducibility work: first, it is perfect, so extensions of trivial modules split, and second it has a Casimir element that separates the trivial module from others. The proof works for any other Lie algebra with these two properties, which we will later see includes all finite dimensional semisimple complex Lie algebras. For SL_2 the Casimir operator distinguishes any two non-isomorphic finite dimensional representations, but this is no longer true for higher rank Lie algebras: there can be several different irreducible representations with the same eigenvalue for the Casimir operator. However there are "higher Casimir" operators in the center of the universal enveloping algebra that separate all finite dimensional irreducible representations.

Complete reducibility is quite rare: in general it fails for infinite dimensional Lie algebras, or simple Lie algebras in positive characteristic, or infinite dimensional representations of simple complex Lie algebras.

Exercise 283 Find an infinite dimensional representation of $sl_2(C)$ that is not completely reducible. Find a perfect finite dimensional complex Lie algebra whose finite dimensional representations are not completely reducible.

Exercise 284 Show that the adjoint representation of $sl_n(F_p)$ on $gl_n(F_p)$ is not completely reducible if p divides n.

Exercise 285 Show that if the finite dimensional representations of a finite dimensional Lie algebra over some field are completely reducible, then the Lie algebra is a direct sum of simple Lie algebras.

Exercise 286 Classify the finite dimensional indecomposable representations of the 1-dimensional abelian complex Lie algebra. What does this have to do with Jordan blocks of the Jordan normal form of a matrix?