21 Schur indicator

Complex representation theory gives a good description of the homomorphisms from a group to a special unitary group. The Schur indicator describes when these are orthogonal or quaternionic, so it is also easy to describe all homomorphisms to the compact orthogonal and symplectic groups (and a little bit of fiddling around with central extensions gives the homomorphisms to compact spin groups). So the homomorphisms to any compact classical group are well understood. However there seems to be no easy way to describe the homomorphisms of a group to the exceptional compact Lie groups.

We have several closely related problems:

- Which irreducible representations have symmetric or alternating forms?
- Classify the homomorphisms to orthogonal and symplectic groups
- Classify the real and quaternionic representations given the complex ones.

Suppose we have an irreducible representation V of a compact group G. We study the space of invariant bilinear forms on V. Obviously the bilinear forms correspond to maps from V to its dual, so the space of such forms is at most 1-dimensional, and is non-zero if and only if V is isomorphic to its dual, in other words if and only if the character of V is real. We also want to know when the bilinear form is symmetric or alternating. These two possibilities correspond to the symmetric or alternating square containing the 1-dimensional irreducible representation. So $(1, \chi_{S^2V} - \chi_{\Lambda^2V})$ is 1, 0, or -1 depending on whether V has a symmetric bilinear form, no bilinear form, or an alternating bilinear form. By the formulas for the symmetric and alternating squares this is given by $\int_G \chi g^2$, and is called the Schur indicator. We have just seen that it vanishes if and only if the character of V has a non-real value.

Exercise 274 If G is a finite group of odd order, show that the map taking g to g^2 is a bijection, and deduce that the Schur indicator of any non-trivial irreducible representation is 0. Using the fact that the degree of an irreducible character divides the order of the group, show that the number of elements of G is equal to the number of conjugacy classes mod 16.

Exercise 275 Find the Schur indicators of the irreducible representations of $SU(2)$.

Lemma 276 *For an irreducible representation V of a compact group the following conditions are equivalent*

- *V has Schur indicator 1*
- *V has has a nonzero invariant symmetric bilinear form*
- *V has an invariant real form*
- *V has an invariant antilinear involution*
- *V is reducible as a real representation.*

Proof It is obvious that real forms correspond to (fixed points of) antilinear involutions. If V has a real form W, then a non-zero invariant symmetric bilinear form of W (which always exists as G is compact: take an average of any positive definite form) gives one on V. Conversely if V has a non-zero invariant bilinear form (a, b) then we can normalize it so that $\Re(a, a) \leq \langle a, a\rangle$ with equality holding for some non-zero a. Then the a for which equality holds form a subspace (as it is the kernel of a positive semidefinite real bilinear form). This subspace is a real form of V. $\square$

Lemma 277 *If V is an irreducible representation of a compact group then the following conditions are equivalent:*

- *V has Schur indicator -1*
- *V has a nonzero invariant skew symmetric form*
- *The underlying real representation of V has a quaternionic structure.*

Proof The existence of a nonzero invariant bilinear form on V is equivalent to the existence of an invariant antilinear map j on V, by putting $(a, b) = \langle a, bj\rangle$. We consider the real algebra generated by i and j. This is either the quaternions or the 2 by 2 matrices over the reals, and the latter case corresponds to V being reducible over the reals. We have seen that the case when V is reducible corresponds exactly to the case when the bilinear form is symmetric, so the case when V is quaternionic corresponds exactly to the case when the bilinear form is antisymmetric. $\square$

To summarize, the real irreducible representation can be read off from the complex ones as follows:

- Complex irreducible representations of complex dimension d with real character and Schur index $+1$ give a real irreducible representation of real dimension d as a real form.
- Complex irreducible representations of complex dimension d with non-real character occur in complex conjugate pairs. The underlying real representations of these two complex representations are isomorphic and give an irreducible real representation of dimension $2d$, with a complex structure (or more precisely two different complex structures).
- Complex irreducible representations of complex dimension d with Schur indicator -1 have d even, and the underlying real vector space is an irreducible real representation with a quaternionic structure or real dimension $2d$ divisible by 4.

  In practice real representations are most common and quaternionic ones tend to be rare.

Example 278 The groups D_8 and Q_8 have the same character table. However the 2-dimensional representations behave differently: one is real and the other is quaternionic.

Example 279 We find the real (orthogonal) representations of $SU(2)$. The Schur indicator of the $n+1$-dimensional irreducible complex representation is $(-1)^n$: we can evaluate this either by decomposing the virtual character $q^n + \cdots q^{-n}$ as a linear combination of characters, or by computing the integral

$$\int_{S^1} (q^{2n} + q^{2n-4} + \cdots)(q - q^{-1})^2 \frac{-1}{2 \times 2\pi} dx$$

where $q = e^{ix}$. So the real representations have dimensions 1, 3, 4, 5, 7, 8, 9,.. and the ones of dimension divisible by 4 are quaternionic.

Exercise 280 Let G be a non-cyclic finite group of rotations in 3-dimensional space (so G is dihedral or the rotations of a Platonic solid). Show that its inverse image in S^3 is a group of order $2|G|$ that has an irreducible 2-dimensional complex representation with Schur index -1.

Exercise 281 By the Wedderburn structure theorem, the real group algebra of G is a sum of matrix algreal division algebras, corresponding to the various real irreducible representations of G. Tensoring with $\mathbb{C}$ gives the complex group ring, which decomposes as a sum of matrix algebras over $\mathbb{C}$, corresponding to the complex irreducible representations of $\mathbb{C}$. Show that this is equivalent to the description of the real representations in terms of the Schur indicator.