Let us check directly that this is indeed the right factor for converting integrals of class functions into integrals over the torus. To see this it is easiest to identify $SU(2)$ with the sphere of unit quaternions, so its maximal torus is the circle of quaternions e^{ix}. Two unit quaternions are conjugate if and only if they have the same real part. So to each element $u = e^{ix}$ of the circle S^1 we get a 2-sphere of unit quaternions conjugate to it. The volume of this 2-sphere is $4\pi \sin(x)^2$, so we see that if f is a class function then

$$\int_{S^3} f(g)dg = \frac{4\pi \sin(x)^2}{2} \int_{x \mod 2\pi} f(e^{ix})dx$$

The factor of 2 in the denominator comes from the fact that there are 2 points in S^1 for most conjugacy classes, and is the the order of the Weyl group.

We can now check directly that we have found an orthonormal base for the class functions on S^3, because the functions $u^{n+1} - u^{-n-1}$ form an orthogonal basis for the odd functions on the unit circle.

This shows that the representations we have found are irreducible because their characters have norm 1 (when the measure on the group is normalized to have volume 1). We can also check directly that the representations are irreducible. For example, any nonzero subrepresentation splits as a sum of eigenspaces of the torus S^1, in other words a sum of spaces generated by monomials $x^n y^{n-i}$, and if we are given such a monomial we can recover the other eigenspaces by acting on it with suitable elements of $SU(2)$ and then taking eigenspaces again.

We will restate the relation between representations of S^3 and its torus in a way that generalizes to compact groups:

- Conjugacy classes of S^3 correspond to elements of the torus modulo the action of the Weyl group.
- The irreducible representations of the torus form a lattice acted on by the Weyl group.
- The irreducible representations of the compact group correspond to orbits of the Weyl group on representations of the torus on which the Weyl group acts faithfully.
- There is a fudge factor relating integration of class functions on the group to integration of Weyl-invariant functions on the torus.
- The character of a representation is an alternating sum over the Weyl group divided by the square root of the fudge factor. (It seems to be a lucky coincidence that the fudge factor for S^3 happens to be a square: in fact this always happens, essentially because the roots of a Lie group come in opposite pairs.)
- Orthogonality for characters of S^3 reduces to orthogonality of characters on the torus. Completeness of characters of S^3 reduces to completeness for functions on the torus that are alternating under the action of the Weyl group.

Example 268 We can decompose tensor products and symmetric squares of representations by calculating characters. For example, the tensor product of the 6 and 3 dimensional representations is the sum of the 4, 6, and 8 dimensional representations, and the symmetric square of the 6-dimensional representation is the sum of the 9, 5, and 1-dimensional representations.

The characters of $SU(2)$ are unimodal. This means the coefficient of q^n is at least that of q^{n+2} whenever $n \geq 0$. They are also symmetric under changing q to q^{-1}. Unimodal polynomials often turn out to be characters of representations of $SU(2)$.

Example 269 Show that the Hopf manifold $C^2/(x, y) = (2x, 2y)$ is not Kaehler. The underlying topological space is $S^1 \times S^3$, so the character of its cohomology ring is $1 + q + q^3 + q^4$ which is not unimodal: there is a gap at q^2. However they theory of Kaehler manifolds shows that the cohomology is a representation of $SU(2)$ with the various cohomology groups corresponding to the eigenspaces of a torus. So the character of the cohomology ring has to be a character of $SU(2)$ (shifted by a power of q).

Exercise 270 The Gaussian binomial coefficients (or q-binomial coefficients) are given by

$$\frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k) \times (1-q)(1-q^2)\cdots(1-q^{n-k})}$$

Show that they are (up to a power of q) characters of $SU(2)$ (in fact the k exterior power of the n-dimensional irreducible character), and therefore unimodal.

We show that we can find any irreducible representation V inside the regular representation. For this we pick any nonzero vector w in the dual of V. Then we can map any vector $v \in V$ to the function on G taking g to $w(g(v))$. This embeds the representation V into the group ring, so we can find all irreducible representations by decomposing the regular representation. In fact we can do better than this. Instead of fixing w we can do this simultaneously for all w in the dual, and we get a linear map from $V \otimes V* = \mathrm{End}(V)$ to the group ring. Moreover the group ring is not just a representation of G, but of $G \times G$, because G can act by multiplication on either the left or the right and these two actions commute.

So for each irreducible representation V we can find an image of $\mathrm{End}(V)$ inside the group ring. There are two natural questions: is the map from $\mathrm{End}(V)$ to the group ring injective, and is the group ring the direct sum of these spaces? Over general fields the answer to both questions is no. For example, over the real numbers this fails even for the cyclic group of order 3 which has irreducible representations of dimensions 1 and 2, and over algebraically closed fields of positive characteristic dividing the order of G it fails badly because the group ring of G has a large radical (a nilpotent ideal). However it does hold over the complex numbers: the group ring is the direct sum of matrix rings of irreducible representations.

Example 271 In particular this shows that the sum of the squares of the irreducible complex representations of a finite group is the order of the group.

This sometimes gives a quick way to check that we have found all irreducible representations.

Remark 272 This generalizes to compact (Lie) groups when it is called the Peter-Weyl theorem: the space of L^2 functions on a compact group splits as a direct sum of spaces naturally isomorphic to the endomorphism rings of the irreducible representations. In general the representation theory of compact Lie groups is rather similar to the representation theory of finite groups. For non-compact groups the result can fail drastically: there are sometimes irreducible unitary representations that cannot be seen inside the regular representation, and this is one reason why finding the irreducible unitary representations of a non-compact Lie group can be rather hard. (In technical terms the support of the Plancherel measure need not be the whole space of unitary irreducible representations.)

Remark 273 This is all closely related to the Wedderburn structure theorem for finite-dimensional semisimple algebras over a field. This says that such an algebra is a direct sum of matrix algebras over division rings. The group ring of a finite group over a field of characteristic 0 is semisimple so splits as a direct sum of matrix algebras over division rings, where each matrix algebra $M_n(D)$ corresponds to an irreducible representation of dimension $n \dim(D)$ where the algebra of endomorphisms commuting with G is the division algebra D. So the order of the group is the sum of numbers n^2d corresponding to representations of dimension nd. Over the complex numbers the division algebras D are all just the complex numbers and the numbers d are all 1.