We will need to use duals of representations. These can get rather confusing because there are in fact 8 different natural ways of constructing a new representation from an old one, many of which have been called the dual. The three main ways are as follows:

- The complex conjugate of a representation V: keep the same G-action, but change the action of i to $-i$. If we represent the elements of G by complex matrices, this corresponds to taking the complex conjugate of a matrix.

- Change the left action on V to a right action. If we define $vg = gv$ with does not work (why?) but we get a right action by putting $vg = g^{-1}v$. (This is really using the antipode of the group ring of G thought of as a Hoof algebra: any left module over a Hopf algebra can be turned into a right module in a similar way.) This corresponds to taking inverses of a matrix.

- The usual vector space dual of V is a representation, but we have to be careful how we define the G-action. Putting $(gf)(v) = f(gv)$ fails. However we can make the dual into a right G-module by putting $(fg)(v) = f(gv)$. This operation corresponds to taking transposes of a matrix.

By combining these three operations in various ways we can construct other representations. For example if we want the dual to be a left module, we first construct the dual as a right module, then change it to a left module, so we use the transpose inverse of matrices. If we want the Hermitian dual as a right module we take the complex conjugate of the dual; if we like we can turn this into a left module by taking inverses as well. (Physicists like to leave it as a right module in their bra-ket notation, while mathematician like to make it into a left module.)

For finite (and compact) groups all irreducible complex representations are unitary (or rather can be made unitary in a way that is unique up to multiplication by non-zero scalars). If we work with unitary representations then taking conjugate inverse transpose leaves everything fixed there are only 4 things we can do, and if we stick to left modules this leaves just two things: V and its dual, which can be given by taking complex conjugates. However if we work with non-unitary representations then there are still 4 left modules we can construct from V and taking ordinary or Hermitian duals is no longer the same as taking complex conjugates.

Example 254 The G to be the circle group $\mathbb{R}/2\pi\mathbb{Z}$. Then its irreducible representations are 1-dimensional and are given by $x \mapsto e^{inx}$ for integers n. There functions form an orthonormal base for the L^2 functions on G (using Lebesgue measure divided by 2π), and in particular every L^2 function can be written as a linear combination of them: this is just its Fourier series expansion. In this case the dual of a representation is given by the complex conjugate, as we expect for compact groups.

Exercise 255 Show that the representations of a finite abelian group give an orthonormal basis for the functions on the group in a similar way.

We would like to generalize this to all finite (and compact) groups: in other words find an orthonormal basis for functions on G related to the irreducible representations.

Lemma 256 *(Schur's lemma). Suppose V and W are irreducible representations of a group G over some field k. Then the algebra of linear transformations of V that commute with G is a division algebra over k. The space of linear transformations commuting with G from V to W is 0 if V is not isomorphic to W.*

Proof This is almost trivial: suppose T is any endomorphism commuting with G. Then the image and kernel of T are invariant subspaces, so must be 0 on the whole space. So either T is 0, or it has zero kernel so is an isomorphism and has an inverse. □

For complex representations the only finite dimensional division algebra over $\mathbb{C}$ is $\mathbb{C}$, so the space of linear maps from V to itself is 1-dimensional. Over other fields more interesting things can happen:

Example 257 If G is the group of order 4 acting on the real plane by rotations, then the algebra of endomorphisms commuting with is is the algebra of complex numbers. This also gives an example of a representation that is irreducible but not absolutely irreducible: it becomes reducible over an algebraic closure.

Example 258 If G is the quaternion group of order 8 acting by left multiplication on the quaternions (thought of as a 4-dimensional real vector space) then the algebra commuting with it is the algebra of quaternions (acting by right multiplication in itself).

If G is a finite group we can construct its group ring $\mathbb{C}[G]$: this is the complex algebra with basis G and multiplication given by the product of G. Alternatively we can think of it as functions on G with the product given by convolution: this definition generalizes better to Lie groups. The regular representation of G is the action of G on its group algebra by left multiplication.

The functions $\langle gv, w\rangle$ are called matrix coefficients of the representation: if we choose a basis for V and the dual basis for W they the endomorphisms of V are given by matrices, and the representation of G is given by matrix-valued functions of G. We will now show that these matrix coefficients are mostly orthogonal to each other under the obvious inner product on $\mathbb{C}[G]$ where the elements of G form an orthonormal base.

Lemma 259 *Suppose the representation V does not contain the trivial representation. Then $\sum_g \langle gv, w\rangle = 0$ for any $v \in V$, $w \in V^*$.*

Proof The vector $\sum_g gv$ is fixed by G and is therefore 0, so has bracket 0 with any w. □

Lemma 260 *If the irreducible representations V and W are not dual, then matrix coefficients of V are orthogonal to matrix coefficients of W under the symmetric inner product on $\mathbb{C}[G]$.*

Proof By assumption $V \otimes W$ does not contain the trivial representation (using Schur's lemma) so

$$\sum_{g \in G} \langle g(a), c\rangle \langle g(b), d\rangle = \sum_{g \in G} \langle g(a \times b), c \times d\rangle = 0$$

□

The character of the dual of a unitary representation is given by taking the complex conjugate, so we get:

Lemma 261 *If V and W are irreducible and not isomorphic, then the matrix coefficients of V and W are orthogonal under the hermitian inner product of $\mathbb{C}[G]$.*

Exercise 262 If V is an irreducible complex representation with some basis show that the sum of matrix coefficients $\sum_{g \in G} g_{ij} g^{-1}_{kl}$ is $|G|/\dim(V)$ if $i = l$, $j = k$ and 0 otherwise. (This is similar to the proof when we have two different representations, except that now there is a non-trivial map from V to V that makes some of the inner products of matrix coefficients non-zero.)

We give the group ring the Hermitian scalar product such that the elements of G are orthogonal and have norm $1/|G|$. To summarize: if we take a representative of each irreducible representation of G and take an orthonormal base of each representation, then the matrix coefficients we get form an orthogonal set in the group ring of G. The norms are given by $1/\dim$ (if we normalize the measure on G so that G has measure 1: this generalizes to compact groups).

Definition 263 *The character of a representation is the function from G to $\mathbb{C}$ given by the trace.*

The character is just the sum of the diagonal entries of a matrix, so by the orthogonality for matrix coefficients we see that the characters of irreducible representations form an orthonormal set of irreducible functions on G. Characters are rather spacial functions on G because they are class functions: this means they only depend on the conjugacy class of an element of G (which follows from the fact that matrices $g^{-1}hg$ and h have the same trace).

Exercise 264 If V and W are representations, show that $V \otimes W$ is a representation whose character is the product of the characters of V and W.

Exercise 265 If G acts on a set S, form a representation of G on the vector space with basis S. Show that the character of this representation is given by taking the number of fixed points of an element of G. Show that this representation always contains the trivial 1-dimensional representation as a subrepresentation. How many times does the trivial 1-dimensional representation occur?

Exercise 266 Show that the character of the symmetric square of a representation with character χ is given by $(\chi(g)^2 + \chi(g^2))/2$ and find a formula for the character of the alternating square. (If g has eigenvalues λ_i, then the eigenvalues on the symmetric square or alternating square are $\lambda_i \lambda_j$ for $i \leq j$ or $i < j$.)