20 Representation theory

A representation of a group is an action of a group on something, in other words a homomorphism from the group to the group of automorphisms of something.

The most obvious representations are permutation representations: actions of a group on a (discrete) set. For example the group of rotations of a cube has order 24, and has several obvious permutation representations: it can act on faces, edges, vertices, diagonals, pairs of opposite faces, and so on.

For another example, the group $SL_2(\mathbb{R})$ acts on the real projective line, and on the upper half plane. These sets carry topologies, which we should take into account when defining representations.

Let us try to classify all permutation representations of some group G . First of all, if the group G acts on some set X, then we can write X as the disjoint union of the orbits of G on X, and G is transitive on each of these orbits. So this reduces the classification of all permutation representations to that of transitive permutation representations. Next, given a transitive permutation representation of G on a set X, fix a point $x \in X$ and write G_x for the subgroup fixing x. Then as a permutation representation, X is isomorphic to the action of G on the set of cosets G/G_x. If we choose a different point y, then G_x is conjugate to G_y (using some element of G taking x to y), so we see that transitive permutation representations up to isomorphism correspond to conjugacy classes of subgroups of G.

Exercise 248 Classify the transitive permutation representations of the group S_4 up to isomorphism. (There are such representations on 1, 2, 3, 4, 6, 6, 6, 8, 12, 12, 24 points.)

More generally we can ask for representations on a set preserving some structure, such as a topology, metric, measure, and so on. By far the most important such structure is that of a vector space over a field: such representations are called linear representations, or sometimes just representations. (It was not at all obvious that these were good things to study: people studied permutation representations of finite groups for several decades before Frobenius started the study of linear representations.)

Just as permutation representations can be decomposed into indecomposable (or transitive) ones, we can try to do the same for linear representations. For finite dimensional representations we can obviously write any representation as a sum of indecomposable ones, where "indecomposable" means that it cannot be written as a sum of two non-zero representations. Unlike the case of permutation representations this decomposition need not be unique, even for the trivial group: a vector space can usually be written as a direct sum of 1-dimensional spaces in more than 1 way. In infinite dimensions a representation cannot always be decomposed into indecomposables: the theory of von Neumann algebras of types II and III is all about this phenomenon.

A linear representation is called irreducible if it is non-zero and has no subrepresentations other than 0 and itself. Obviously any irreducible representation is indecomposable, but the converse need not hold.

Example 249 For quivers, the irreducible representations are those of total dimension 1, while the indecomposables can be much more complicated, as in the previous lecture. The representation of the integers on $\mathbb{R}^2$ taking n to $\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$ is indecomposable but not irreducible. The finite group $\mathbb{Z}/p\mathbb{Z}$ acting on the 2-dimensional vector space $\mathbb{F}_2$ by the same formula is indecomposable but not irreducible.

Example 250 Suppose we look at (representations of the trivial group on) finitely generated abelian groups. The indecomposable things are the groups $\mathbb{Z}$ and $\mathbb{Z}/p^n\mathbb{Z}$ for primes p, while the irreducible things are the groups $\mathbb{Z}/p\mathbb{Z}$. In particular there is an indecomposable object that cannot be broken up into irreducibles.

So in general indecomposable things need not be irreducible. The problem is the following: given a subrepresentation $A \to B$ can we always find a complement, in other words a representation C so that B is the sum of A and C?If we can always do this, then obviously indecomposables and irreducibles are the same. There is one very important case when we can always do this: when the representation B is finite dimensional and unitary, in other words the group action preserves a positive definite Hermitian inner product. Then we can just take C to be the orthogonal complement of A. (In infinite dimensions we need to add some further conditions: the space B should be complete, in other words a Hilbert space, and the subspace A should be closed.) This is a major reason why unitary representations are so popular: we do not have to deal with the hard problem of indecomposables that are not irreducible.

In general we say that a representation is completely reducible if it is a sum of irreducible representations.

Theorem 251 *Any finite-dimensional complex representation of a finite group is completely reducible.*

Proof The idea is to show that the representation V of the finite group G is unitary. So choose any old positive definite Hermitian inner product. The problem is that this inner product need not be invariant under the group action. However we can make it invariant under the group action by averaging over the group G: we take all images of the inner product under the action of G and take their average. A slightly subtle point is that this average is still positive definite (and in particular non-zero) which follows from positive definiteness. So the representation has an invariant positive definite inner product and is therefore completely reducible. $\square$

We can use this to classify all finite-dimensional complex representations of a finite abelian group. The irreducible representations are easy to find: they are all 1-dimensional because any commuting operators on a finite-dimensional complex vector space have a common eigenvector. The 1-dimensional representations of a group form a group under tensor product, called the character group.

Exercise 252 Show that if a finite abelian group is a direct sum of cyclic groups of orders a, b, c, ... then its character group can also be written in this form. (This result is a bit misleading: although a finite abelian group and its character group are isomorphic, there is usually no natural isomorphism between them.)

So since we know the irreducible finite-dimensional representations of a finite abelian group, and we know that every finite dimensional representation is a sum of irreducibles, this classifies all finite-dimensional representations of a finite abelian group.

Example 253 We find some irreducible representations of the non-abelian finite group S_3 of order 6. There are two obvious 1-dimensional representations: the trivial one, and the "sign representation" where even and odd elements act as 1 and -1. There is also a 2-dimensional (real) representation: think of S_3 acting on the 3 corners of an equilateral triangle with center 0. These are all the irreducible representations, though to see this we need some more theory.