3. Some Lie groups do not correspond to ANY algebraic group. We give two examples here. There are two ways that a group might not be algebraic: its center might be too small or too large.

   The *Heisenberg group* is the subgroup of symmetries of $L^2(\mathbb{R})$ generated by translations ($f(t) \mapsto f(t+x)$), multiplication by $e^{2\pi ity}$ ($f(t) \mapsto e^{2\pi ity} f(t)$), and multiplication by $e^{2\pi iz}$ ($f(t) \mapsto e^{2\pi iz} f(t)$). The general element is of the form $f(t) \mapsto e^{2\pi i(yt+z)} f(t+x)$. This can also be modeled as

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \Big/ \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$$

   It has the property that in any finite dimensional representation, the center (elements with $x = y = 0$) acts trivially, so it cannot be isomorphic to any algebraic group. In some sense this fails to be algebraic because the center is too small: if we take the universal cover, making the center larger, it becomes algebraic.

   It is somewhat easier to study the representations of its Lie algebra, which has a basis of 3 elements X, Y, Z with $[X, Y] = Z$. We can represent this in infinite dimensions by putting $X = x$, $Y = d/dx$, $Z = 1$ acting on polynomials (this is Leibniz's rule!) In quantum meachanics this algebra turns up a lot where X is the position operator and Y the momentum operator. But it has not finite dimensional representations with Z acting as 1 because the trace of Z must be 0.

   The *metaplectic group*. Let's try to find all connected groups with Lie algebra $\mathfrak{sl}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a + d = 0\}$. There are two obvious ones: $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$. There aren't any other ones that can be represented as groups of finite dimensional matrices. However, if you look at $SL_2(\mathbb{R})$, you'll find that it is not simply connected. To see this, we will use Iwasawa decomposition.

   Theorem 16 (Iwasawa decomposition) *If G is a (connected) semisimple Lie group, then there are closed subgroups K, A, and N, with K compact, A abelian, and N unipotent, such that the multiplication map $K \times A \times N \to G$ is a surjective diffeomorphism. Moreover, A and N are simply connected.*

   In the case of SL_n, this is the statement that any basis can be obtained uniquely by taking an orthonormal basis ($K = SO_n$), scaling by positive reals (A is the group of diagonal matrices with positive real entries), and shearing (N is the group $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$). This is exactly the result of the Gram-Schmidt process.

   The upshot is that $G \simeq K \times A \times N$ (topologically), and A and N do not contribute to the fundamental group, so the fundamental group of G is the same as that of K. In our case, $K = SO_2(\mathbb{R})$ is isomorphic to a circle, so the fundamental group of $SL_2(\mathbb{R})$ is $\mathbb{Z}$.

   So the universal cover $\widetilde{SL_2(\mathbb{R})}$ has center $\mathbb{Z}$. Any finite dimensional representation of $\widetilde{SL_2(\mathbb{R})}$ factors through $SL_2(\mathbb{R})$, so none of the covers of

$SL_2(\mathbb{R})$ can be written as a group of finite dimensional matrices. The universal cover fails to be algebraic because the center is too large.

The most important case is the metaplectic group $Mp_2(\mathbb{R})$, which is the connected double cover of $SL_2(\mathbb{R})$. It turns up in the theory of modular forms of half-integral weight and has a representation called the metaplectic representation. This appears in the theory of modular forms of half integral weight.

p-adic Lie groups are defined in a similar way to Lie groups using p-adic manifolds rather than smooth manifolds. They turn up a lot in number theory and algebraic geometry. For example, Galois groups act on varieties defined over number fields, and therefore act on their (etale) cohomology groups, which are vector spaces over p-adic fields, sometimes with bilinear forms coming from Poincare duality. So we get representations of Galois groups into p-adic Lie groups. These see much more of the Galois group than representations into real Lie groups, which tend to have finite images.

1.4 Lie groups and Lie algebras

Lie groups can have a rather complicated global structure. For example, what does $GL_n(\mathbb{R})$ look like as a topological space, and what are its homology groups? Lie algebras are a way to linearize Lie groups. The Lie algebra is just the tangent space to the identity, with a Lie bracket $[,]$ which is a sort of ghost of the commutator in the Lie group. The Lie algebra is almost enough to determine the connected component of the Lie group. (Obviously it cannot see any of the components other than the identity.)

Exercise 17 The Lie algebra of $GL_n(\mathbb{R})$ is $M_n(\mathbb{R})$, with Lie bracket $[A, B] = AB - BA$, corresponding to the fact that if A and B are small then the commutator $(1 + A)(1 + B)(1 + A)^{-1}(1 + B)^{-1} = 1 + [A, B] +$ higher order terms. Show that $[A, B] = -[B, A]$ and prove the Jacobi identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

An abstract Lie algebra over a field is a vector space with a bracket satisfying these two identities.

We have an equivalence of categories between simply connected Lie groups and Lie algebras. The correspondence cannot detect

- Non-trivial components of G. For example, SO_n and O_n have the same Lie algebra.
- Discrete normal (therefore central) subgroups of G. If $Z \subseteq G$ is any discrete normal subgroup, then G and G/Z have the same Lie algebra. For example, $SU(2)$ has the same Lie algebra as $PSU(2) \simeq SO_3(\mathbb{R})$.

If $\tilde{G}$ is a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, then any other connected group G with Lie algebra $\mathfrak{g}$ must be isomorphic to $\tilde{G}/Z$, where Z is some discrete subgroup of the center. Thus, if you know all the discrete subgroups of the center of $\tilde{G}$, you can read off all the connected Lie groups with the given Lie algebra.

Let's find all the groups with the algebra $\mathfrak{so}_4(\mathbb{R})$. First let's find a simply connected group with this Lie algebra. You might guess $SO_4(\mathbb{R})$, but that isn't simply connected. The simply connected one is $S^3 \times S^3$ as we saw earlier (it is a product of two simply connected groups, so it is simply connected). The center of S^3 is generated by -1, so the center of $S^3 \times S^3$ is $(\mathbb{Z}/2\mathbb{Z})^2$, the Klein four group. There are three subgroups of order 2

Therefore, there are 5 groups with Lie algebra $\mathfrak{so}_4$.

Formal groups are intermediate between Lie algebras and Lie groups: We get maps (Lie groups) to (Formal groups) to (Lie algebras). In characteristic 0 there is little difference between formal groups and Lie algebras, but over more general rings there is a big difference. Roughly speaking, Lie algebras seem to the the "wrong" objects in this case as they do not see enough of the group, and formal groups seem to be a good replacement.

A 1-dimensional formal group is a power series $F(x, y) = x + y + \cdots$ that is associative in the obvious sense $F(x, F(y, z)) = F(F(x, y), z)$. For example $F(x + y) = x + y$ is a formal group called the additive formal group.

Exercise 18 Show that $F(x + y) = x + y + xy$ is a formal group, and chack that over the rationals it is isomorphic to the additive formal group (in other words there is a power series with rational coefficients such that $F(f(x), f(y)) = f(x + y)$). These formal groups are not isomorphic over the integers.

Higher dimensional formal groups are defined similarly, except they are given by several power series in several variables.

1.5 Important Lie groups

Dimension 1: There are just $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z}$.

Dimension 2: The abelian groups are quotients of $\mathbb{R}^2$ by some discrete subgroup; there are three cases: $\mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z} = \mathbb{R} \times S^1$, and $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$.

There is also a non-abelian group, the group of all matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, where $a > 0$. The Lie algebra is the subalgebra of 2×2 matrices of the form $\begin{pmatrix} h & x \\ 0 & -h \end{pmatrix}$, which is generated by two elements H and X, with $[H, X] = 2X$. This is the smallest example of a solvable but non-abelian connected Lie group.

Dimension 3: There are some abelian and solvable groups, such as $\mathbb{R}^2 \ltimes \mathbb{R}^1$, or the direct sum of $\mathbb{R}^1$ with one of the two dimensional groups. As the dimension increases, the number of solvable groups gets huge, so we ignore them from here on.

The group $SL_2(\mathbb{R})$, which is the most important Lie group of all. We saw earlier that $SL_2(\mathbb{R})$ has fundamental group $\mathbb{Z}$. The metaplectic double cover $Mp_2(\mathbb{R})$ is important. The quotient $PSL_2(\mathbb{R})$ is simple, and acts on the open upper half plane by linear fractional transformations

Closely related to $SL_2(\mathbb{R})$ is the compact group SU_2. We know that $SU_2 \simeq S^3$, and it covers $SO_3(\mathbb{R})$, with kernel ± 1. After we learn about Spin groups, we will see that $SU_2 \cong \mathrm{Spin}_3(\mathbb{R})$. The Lie algebra $\mathfrak{su}_2$ is generated by three elements X, Y, and Z with relations $[X, Y] = 2Z$, $[Y, Z] = 2X$, and $[Z, X] = 2Y$. An explicit representation is given by $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The cross product on $\mathbb{R}^3$ gives it the structure of this Lie algebra.

The Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$ are non-isomorphic, but when you complexify, they both become isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

There is another interesting 3 dimensional algebra. The Heisenberg algebra is the Lie algebra of the Heisenberg group. It is generated by X, Y, Z, with $[X, Y] = Z$ and Z central. You can think of this as strictly upper triangular matrices.

Dimension 6: (nothing interesting happens in dimensions 4,5) We get the group $SL_2(\mathbb{C})$. This is the smallest complex simple Lie group. Later, we will see that it is also $\text{Spin}_{1,3}(\mathbb{R})$, the double cover of rotations in Minkowski space, so is used a lot in special relativity.

Dimension 8: We have $SU_3(\mathbb{R})$ and $SL_3(\mathbb{R})$. This is the first time we get a 2-dimensional root system.

Dimension 10: The symplectic group $Sp_2(\mathbb{R})$. This is the first example of a simple group that is not some variation of a special linear group. It has traditionally been the group where people discover phenomena that cannot be seen in general linear groups (such as the existence of cuspidal unipotent representations).

Dimension 14: G_2, the first exceptional simple group. It is more or less the automorphisms of the Octonions, an 8-dimensional non-associative algebra.

Dimension 248: E_8, which gets into the newspapers more than any other Lie group.

This class is mostly about finite dimensional algebras, but let's mention some infinite dimensional Lie groups or Lie algebras.

1. Automorphisms of a Hilbert space form a Lie group.

2. Diffeomorphisms of a manifold form a Lie group. There is some physics stuff related to this.

3. *Gauge groups* are (continuous, smooth, analytic, or whatever) maps from a manifold M to a group G.

4. The *Virasoro algebra* is generated by L_n for $n \in \mathbb{Z}$ and c, with relations $[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m,0}\frac{n^3-n}{12}c$, where c is central (called the *central charge*). If you set $c = 0$, you get (complexified) vector fields on S^1, where we think of L_n as $ie^{in\theta}\frac{\partial}{\partial\theta}$. Thus, the Virasoro algebra is a central extension

$$0 \to c\mathbb{C} \to \text{Virasoro} \to \text{Vect}(S^1) \to 0.$$

5. Affine Kac-Moody algebras, which are more or less central extensions of certain gauge groups over the circle.