19 Quivers and tilting

We will describe an unexpected connection between representations of quivers and simple Lie algebras. To summarize, the quivers with a finite number of indecomposable representations correspond to certain semisimple Lie algebras, the indecomposable representations correspond to positive roots, and the irreducible representations correspond to simple roots.

Definition 241 *A quiver is a finite directed graph (possible with multiple edges and loops). A representation of a quiver (over some fixed field) consists of a vector space for each vertex of the graph and a linear map between the corresponding vector spaces for each edge.*

Example 242 Representations of a point are just vector spaces. Representations of a point with a loop are vector spaces with an endomorphism. Over an algebraically closed field the indecomposable representations are classified by Jordan blocks. Representations of 2 points joined by a line are just linear maps of vector spaces. There are 3 indecomposable representations: a map from a 0-dimensional space to a 1-dimensional one, a map from a 1-dimensional space to a 0-dimensional one, and a map from a 1-dimensional space onto a 1-dimensional one. More generally, stars with n incoming arrows correspond to n maps to a vector space. When $n = 2$ there are 6 indecomposable representations, and when $n = 4$ there are 12. When $n = 4$ there is a qualitative change: there are now infinitely many indecomposables. For example we can take 4 1-dimensional subspaces of a 2-dimensional space. The first two determine a base $(0, 1)$ and $(1, 0)$, the third is spanned by $(1, 1)$ and determines the ratio between the two bases, but nor the 4th space can be spanned by $(1, a)$ for any a, so we get a 1-parameter family of indecomposables. Although there are an infinite number of indecomposables, it is not hard to classify them explicitly: it is a "tame" problem. For stars with 5 incoming vertices the indecomposable representations are "wild": there is no neat description of them. We will see that the cases with a finite number of indecomposables correspond to Dynkin diagrams of the finite dimensional semisimple Lie algebras with all roots the same length, and the tame cases correspond to affine Dynkin diagrams.

The representations of a quiver are the same as modules over a certain ring associated with the quiver. This ring has an idempotent for each vertex, with the idempotents commuting and summing to 1. There is also an element for each edge, subject to some obvious relations. The algebra is finite dimensional if the quiver contains no cycles.

There are two titling functors we can apply to modules over quivers:

- If a vertex a is a source, then we can change all the arrows to point into a, and change the vector space of a to be $Coker(V_a \mapsto \oplus_{a\to b} V_b)$.
- If a vertex a is a sink, then we can change all the arrows to point out of a, and change the vector space of a to be $Ker(\oplus_{b\to a} V_b \mapsto V_a)$

The functors take representations of a quiver to representations of a different quiver, with a source changed to a sink or a sink changed to a source. They are almost but not quite inverses of each other. They are inverses provided

$V_a \mapsto \oplus_{a\to b} V_b$ is injective, or $\oplus_{b\to a} V_b \mapsto V_a$ is surjective. In particular they are inverses of each other on indecomposable modules, except for the special case of indecomposable modules of total dimension 1.

The idea is that we try to classify irreducible modules by repeated applying tilting functors, trying to make the module vanish. If we succeed then we can recover the original module from a 1-dimensional module by applying the "almost inverse" tilting functors in the opposite order. We will see that we can do this provided the quiver is one of the diagrams A_n, D_n, E_6, E_7, and E_8,

Take a vector space spanned by the vertices of a quiver, and give it an inner product such that the vertices of a quiver have norm 2, and their inner product is minus the number of lines joining them. Then the dimension vector of a quiver can be represented by a point in this space in the obvious way, and the effect of tilting by a source or sink a is just reflection in the hyperplane $a^\perp$ (except on the vector a itself).

We want to find a sequence of tiltings so that the dimension vector has a negative coefficient. It is easy enough to find a sequence of reflections of simple roots that do this: the problem is that we have a constraint that we can only use a reflection of a simple root if it is a source or a sink for the quiver. To do this we will use Coxeter elements.

A Coxeter element of a reflection group is a product of the reflections of simple roots in some order.

Lemma 243 *If a Coxeter element of a reflection group fixes a vector then the vector is orthogonal to all simple roots..*

Proof If a vector $a = \sum a_n v_n$ is fixed by a Coxeter element (for simple roots v_i) then the reflection of v_i is the only one that can change the coefficient of v_i, so it must fix a. So a is fixed by all reflections of simple roots, and is therefore orthogonal to all simple roots, so is 0. □

Corollary 244 *If σ is a Coxeter element of a finite reflection group and a is a non-zero vector, then $\sigma^k(a)$ has a negative coefficient for some k.*

Proof Otherwise we could find a non-zero fixed vector $a + \sigma(a) + \sigma^2(a) + \cdots + \sigma^{h-1}(a)$, where h is the order of the Coxeter element. □

Exercise 245 Show that if a Coxeter diagram of a reflection group is a tree then any two Coxeter elements are conjugate, and in particular have the same order (called the Coxeter number).

Exercise 246 Find the order of the Coxeter elements of A_n.

We now construct a special Coxeter element σ associated to a given quiver as follows. First take the reflection of some source, change the source to a sink, and then mark that vertex as used. Keep repeating this until all vertices have been used. The result is the original quiver, as each edge has had its direction changed twice. So we have found a sequence of reflections of sources that preserves the quiver. This means that we can keep on repeating the sequence of reflections of the Coxeter element, and every time we will be reflecting in some source.

We can now show that the indecomposable representations of any quiver of type A_n, D_n, or E_n correspond to the positive roots of the associated root system: in fact we can apply tiltings until the dimension vector becomes a simple root, when it is trivial to find the unique indecomposable. Take the dimension vector a of any indecomposable representation. As $\sigma^k(a)$ has negative coefficients for some k, we can find a finite sequence of tiltings so that some coefficient of the dimension vector becomes negative, which means that there is some sequence of tiltings reducing the dimension vector to a simple root. In particular the dimension vector must have been a positive root, and there is a unique indecomposable representation with this dimension vector (given by applying the sequence of tiltings in reverse order to the representation corresponding to a simple root).

For affine root systems this argument fails but only just: the inner product space spanned by simple roots has a 1-dimensional subspace that has inner product 0 with all vectors, and the dimension vector of an indecomposable is either conjugate to a simple root by a series of tiltings, or is in this 1-dimensional subspace. We saw an example of the latter for the root system of affine D_4.

Exercise 247 If a quiver contains a cycle show that it has an infinite number of inequivalent indecomposable representations. Show more precisely that there are an infinite number of dimension vectors corresponding to indecomposable representations, and (over infinite fields) there is a dimension vector with an infinite number of corresponding indecomposable representations.

For the affine diagrams of types A_n, D_n, E_6, E_7, E_8 there are an infinite number of indecomposables, but their classification is tame, meaning roughly that it can be described explicitly. The dimension vectors just correspond to the positive roots of affine Kac-Moody algebras. For non-affine diagrams the classification is wild and very hard to describe. For example for a point with two loops the representations are just pairs of matrices acting on a vector space, which are notoriously hard to classify.