18 Root systems and reflection groups

We have seen that simple complex Lie algebras have a root system associated to them: this means a finite set of non-zero vectors in Euclidean space, called roots, such that r and s are roots then (r, s) is an integer multiple of $(r, r)/2$ and the reflection $s - 2r(r, s)/(r, r)$ of s in $r^\perp$ is also a root. In particular for every root system we have a Weyl group generated by reflections. The correspondence between semisimple complex Lie algebras and root systems is not quite 1:1 because the root system of a semisimple complex Lie algebra is also reduced: this means that it r is a root then $2r$ is not a root. An example of a non-reduced root system is BC_n, consisting of the vectors $\pm x_i$, $\pm x_i \pm x_j$, and $\pm 2x_i$. In fact these are the only irreducible non-reduced root systems. They are in fact roots sytems of finite dimensional simple superalgebras. Most but not all reflection groups in Euclidean space turn up as Weyl groups of Lie groups: the exceptions are most dihedral groups, the symmetries of an icoshedron, and a group in 4-dimensions.

The classification of root systems and reflection groups is similar. We will do the case of root systems all of whose roots have the same length; the general case uses no essentially new ideas.

The first step is to consider the Coxeter diagram of a reflection group, or the Dynkin diagram of a root system. These are defined as follows. Chop up space by cutting along the reflection hyperplanes. The closed regions bounded by these hyperplanes are called Weyl chambers. Any Weyl chamber is conjugate to its neighbors by a reflection, so all Weyl chambers are conjugate by elements of the reflection group. We pick one Weyl chamber. The Coxeter diagram has a point for each face of the Weyl chamber, and lines between the points according to the angle between faces. If the faces are orthgonal there are no lines, if the faces have an angle of $\pi/3$ there is 1 line, if the faces have an angle of $\pi/4$ there are two lines, and for other angles conventions vary. A Dynkin diagram of a root system is like a Coxeter diagram, with some extra information to indicate the lengths of the roots (which is not defined for arbitrary reflection groups). Usually one draws an inequality sign on the lines to indicate the longest of each pair of roots. (For finite root systems this gives enough information to reconstruct the root system, though for infinite root systems one sometimes needs more information.)

We will work out the Dynkin diagrams of the most of the root systems we have seen.

- A_n: simple roots $\alpha_i - \alpha_{i+1}$
- B_n: simple roots $\alpha_i - \alpha_{i+1}$, α_n
- C_n: simple roots $\alpha_i - \alpha_{i+1}$, $2\alpha_n$
- D_n: simple roots $\alpha_i - \alpha_{i+1}$, $\alpha_{n-1} + \alpha_n$
- E_8 $\alpha_i - \alpha_{i+1}$, $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - \alpha_6 - \alpha_7 - \alpha_8)/2$

  Exercise 236 what happens if there are 4, 6, 7, or 8 minus signs in this last simple root? Why do theese not give Dynkin diagrams for the E_8 root system?

- F_4: $\alpha_1 - \alpha_2$, $\alpha_2 - \alpha_3$, α_3, $-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2$.
- G_2:

A_n, B_n, C_n, D_n, G_2
These Dynkin diagrams explain the numerous local isomorphisms between small simple Lie groups.

- The Dynkin diagrams of A_1, B_1, and C_1 are all points, corresponding to the fact that SL_2, $O_{2,1}$, and Sp_2 are locally isomorphic.
- The Dynkin diagrams of B_2 and C_2 are isomorphic, corresponding to the local isomorphism of SP_4 and $O_{3,2}$.
- The Dynkin diagrams of B_n and C_n are almost the same, explaining why the Lie groups of Sp_{2n} and O_{2n+1} have the same dimension and the same Weyl group.
- The Dynkin diagram of D_2 is two points, corresponding to the fact that $O_{2,2}$ locally splits as $SL_2 \times SL_2$.
- The Dynkin diagram of D_3 is isomorhi9c to A_3, corresponding to the local isomorphism of SL_4 and $O_{3,3}$.
- The Dynkin diagram of D_4 has an extra automor0phism of order 3, corresponding to triality of $Spin_8$.
- The Dynkin diagrams of A_n and D_n have automorphisms of order 2, corresponding to outer automorphisms of SL_{n+1} and O_{2n} given by matrices of determinant -1. (Similarly E_6 has an outer automorphism.)

Coxeter diagrams also make sense for reflection groups not associated to Lie groups, and for infinite reflection groups in Euclidean or hyperbolic space,

Example 237 The Coxeter diagram of the rotations of an icosahedron is

Example 238 The group $GL_2(\mathbb{Z})$ is a reflection group acting on the upper half plane (considered as a quotient of the non-real complex numbers by complex conjugation) and is therefore a hyperbolic reflection group. A Weyl chamber consists of the complex numbers with $0 \leq \Re(z) \leq 1/2$, $|z| \geq 1$, so the Coxeter diagram is (Notice that two of the sides only meet at infinity, so the angle between them is zero.)

Example 239 Conway found the following stunning example of a Dynkin diagram. The 26 dimensional even Lorentzian lattice $II_{1,25}$ is acted on by a hyperbolic reflection group generated by the reflectins of its norm -2 vectors. The Dynkin diagram of this reflection group is the affine Leech lattice. The reaction of many mathematicians to this statement is to regard it as nonsense, on groups that a Dynkin digram is a graph, not a lattice in Euclidean space. However the Dynkin diagram is really the set of simple roots of a root system, and in particular is a metric space. (The graph is just a convenient way of describing this metric space). The Leech lattice is also a metric space, and Conway showed these two metric spaces are isometric. The isometry can be described explicitly