17 Cayley numbers and G2

Definition 230 *A (possibly nonassociative) algebra is called alternative if for all a and b we have $(aa)b = a(ab)$ and $b(aa) = (ba)a$.*

Alternative algebras have the apparently more general property that the subalgebra generated by any two elements is associative.

The Cayley-Dickson construction turns an algebra into another one of twice the dimension as follows. Suppose that A is a module over some commutative ring with a bilinear product (possibly nonassociative and non-commutative) and an involution $*$ with $(ab)^* = b^*a^*$, $a^{**} = a$. For $\gamma \in R$ we define an involution and product on $A \oplus A$ by

$$(p, q)(r, s) = (pr - \gamma s^* q, sp + qr^*)$$

$$(p, q)^* = (p^*, -q)$$

We will assume that A has an identity 1 and that $a + a^*$ and aa^* are always multiples of 1. Then the same is true of $A \oplus A$. Moreover if A is commutative and $* = 1$ then $A \oplus A$ is commutative, and if A is commutative and associative then $A \oplus A$ is associative, and if A is associative then $A \oplus A$ is alternative.

Starting with the real numbers, and taking γ to be 1 at every step, we get the real numbers, the complex numbers, the quaternions, and the octonions (or Cayley numbers), an 8-dimensional alternative algebra over the reals. We could of course continue further to get algebras of dimension 16, 32, and so on, but it seems to be rather hard to think of anything interesting to say about them.

We define the norm $N(a)$ to be $a^*a = aa^*$. When the algebra A is alternative, we have $N(ab) = N(a)N(b)$, and in particular the norms of elements are closed under multiplication. If $N(a)$ is invertible in R then a has an inverse $a^{-1} = a/N(a)$, with $a^{-1}(ab) = b = (ba)a^{-1}$. In particular the octonions form a nonassociative division algebra.

Exercise 231 In a commutative ring, show that the set of sums of n squares is closed under multiplication if n is 1, 2, 4, or 8. (There are no other positive integers for which this is true for all commutative rings.)

If we apply this construction to the complex numbers we get the quaternions, with a basis $1, i, j, k$ and products $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

Exercise 232 Show that the manifold of octonions of norm 1 and trace 0 has an almost complex structure (in other words each tangent space can be made into a complex vector space in a smooth way) and is diffeomorphic to the sphere S^6. (The spheres S^0, S^2, and S^6 are the only spheres with almost complex structures. It is not known whether the sphere S^6 has a complex structure.)

Now we investigate the automorphism group G of the octonions. It is compact as it is a closed subgroup of the orthogonal group in 8 (or even 7) dimensions. We start finding an upper bound on its dimension. The group G acts on the octonions orthogonal to 1, and preserves norms, so it acts on the sphere S^6 of imaginary octonions of norm 1. Now look at the subgroup H fixing some point i of this sphere. In turn H acts on the sphere of all points of norm 1 orthogonal to 1 and i, which is a 5-sphere S^5. Fix j in this 5-sphere, with i and j generating a copy of the quaternions. Now look at the group fixing i and j. This acts on the sphere S^3 of unit octonions orthogonal to $1, i, j, ij$. Since the octonions are generated by i, j, and a point of this 3-sphere, we see that G has dimension at most $3 + 5 + 6 = 14$.

We want to show that G has exactly dimension 14, so we need to construct enough elements of G to show that the various actions on spheres above are transitive, in other words we need to find some automorphisms of the Cayley numbers. An automorphism is given by mapping (p, q) to (p, rq) for a suitable quaternion: if we look at the formula for multiplication we see that this is an automorphism of the octonions provided that r has norm 1.

The octonions have a basis $e_0 = 1, e_1, \cdots, e^7$ such that $e_1e_2 = e_4$ and cyclic permutations of $(1, 2, 3, 4, 5, 6, 7)$ are automorphisms of the Cayley algebra. We can identify the subspace spanned by $1, e_1, e_2, e_4$ with a copy of the quaternions. We will show that this automorphism of order 7, together with the automorphisms of the form $(p, q) \to (p, rq)$ generate the full automorphism group. In fact we will just use this 3-dimensional group and its 6 conjugates under elements of the cyclic group of order 7. In particular the group of automorphisms fixing e_1, e_2, e_4 acts transitively on the norm 1 vectors that are linear combinations of e_3, e_5, e_6, e_7, and similarly for all cyclic permutations.

We show that the automorphism group acts transitively on the sphere S^6 of norm 1 imaginary octonions. Pick a vector in this sphere. We can first kill off the coefficients of e_5, e_6, e_7. Then by acting with the group fixing e_6, e_7, e_2 we can keep the coefficients of e_6, e_7 zero and kill off the coefficients of e_3, e_4, e^5. Then by acting with the group fixing e_4, e_6, e_7 we can make the vector equal to e_1. So G acts transitively on norm 1 imaginary octonions. .

Now we show that the subgroup fixing e_1 acts transitively on the sphere S^5 of imaginary norm 1 octonions orthogonal to e_1 in a similar way. We first act with the group fixing e_1, e_2, e_4 to kill off the coefficients of e_5, e_6, e_7. Then act with the group fixing e_7, e_1, e_3 to kill off the coefficients of e_4, e_5, e^6, keeping the coefficient of e_7 zero. Finally we act with the group fixing e_5, e_6, e_1 to move the element to e_2.

The subgroup fixing the two vectors e_1 and e_2 (and therefore also $e_1e_2 = e_4$, acts transitively on the sphere S^3 of norm 1 imaginary octonions orthogonal to these three vectors. So we have verified that the group G_2 of automorphisms of the Cayley numbers has dimension exactly 14.

We can read off more information about G_2. We found subgroups A, B with $G/A = S^6$, $A/B = S^5$, $B = S^3$. From this we see that G_2 is connected, simply connected, and (using the exact sequence of homotopy groups of a fibration) the second homotopy group vanishes, and the third is Z.

Obviously any automorphism of the quaternions gives an automorphism of the octonions $H \oplus H$ as the construction of the octonions from the quaternions is functorial.

Exercise 233 There is an obvious guess for a maximal integral form for the octonions, analogous to the Hurwitz quaternions, which is to take the octonions whose coordinates are integers or half integers, such that the set with half integer parts is empty or the image of $(\infty 124)$ under $Z/7Z$ or the complement of one of these sets. However this does not work (and is a common error). Show that this integral form is not closed under multiplication. Show that if we swap ∞ and

n in the set of allowed integral parts for some fixed $n \in \mathbb{Z}/7\mathbb{Z}$ (where $1 = e_\infty$) we get an integral form of the octonions that is closed under multiplication. Show that the lattice we get is the E_8 lattice up to rescaling, and there are 240 integral vectors of norm 1.

Exercise 234 Show that the compact group G_2 is the group of automorphisms of R^7 (with basis $x_0, x_1, \ldots, x_6$) preserving the element

$$x_0 \wedge x_1 \wedge x_3 + x_1 \wedge x_2 \wedge x_4 + x_2 \wedge x_3 \wedge x_5 + x_3 \wedge x_4 \wedge x_6 + x_4 \wedge x_5 \wedge x_0 + x_5 \wedge x_6 \wedge x_1 + x_6 \wedge x_0 \wedge x_2$$

of $\wedge^3(R^7)$. The can be thought of as a sum over the 7 lines of the 7-point projective plane of the field with 2 elements.

Exercise 235 Find a Cartan subalgebra and the root system of G_2. (Start by looking at the subgroup fixing a point of S_6. This is isomorphic to $SU(3)$, of dimension 8, so gives the Cartan subgroup and 6 of the roots arranged as the vertices of a hexagon. If the hexagon is written as 6 triangles, the remaining 6 roots are at the centers of the 6 triangles, so there are 6 short roots and 6 long roots that are $\sqrt{3}$ times as long.)