15.3 More about Orthogonal groups

Is $O_V(K)$ a simple group? NO, for the following reasons:

(1) There is a determinant map $O_V(K) \to \pm 1$, which is usually onto. (In characteristic 2 there is a similar Dickson invariant map).

(2) There is a spinor norm map $O_V(K) \to K^\times/(K^\times)^2$

(3) $-1 \in$ center of $O_V(K)$.

(4) $SO_V(K)$ tends to split if $\dim V = 4$, is abelian if $\dim V = 2$, and trivial if $\dim V = 1$.

(5) There are a few cases over small finite fields where the orthogonal group is solvable, such as $O_3(\mathbb{F}_3)$.

It turns out that they are simple apart from these reasons why they are not. Take the kernel of the determinant, to get to SO, then take the elements of spinor norm 1, then quotient by the center, and assume that $\dim V \geq 5$. Then this is usually simple, except for a few small cases over small finite fields. If K is a finite field, then this gives many finite simple groups.

Note that $SO_V(K)$ is NOT the subgroup of $O_V(K)$ of elements of determinant 1 in general; it is the image of $\Gamma_V^0(K) \subseteq \Gamma_V(K) \to O_V(K)$, which is the correct definition. Let's look at why this is right and the definition you know is wrong. There is a homomorphism $\Gamma_V(K) \to \mathbb{Z}/2\mathbb{Z}$, which takes $\Gamma_V^0(K)$ to 0 and $\Gamma_V^1(K)$ to 1 (called the Dickson invariant). It is easy to check that $\det(v) = (-1)^{\text{Dickson invariant}(v)}$. So if the characteristic of K is not 2, $\det = 1$ is equivalent to Dickson $= 0$, but in characteristic 2, determinant is a useless invariant (because it is always 1) and the right invariant is the Dickson invariant.

Special properties of $O_{1,n}(\mathbb{R})$ and $O_{2,n}(\mathbb{R})$. $O_{1,n}(\mathbb{R})$ acts on hyperbolic space $\mathbb{H}^n$, which is a component of norm -1 vectors in $\mathbb{R}^{n,1}$. $O_{2,n}(\mathbb{R})$ acts on the "Hermitian symmetric space" (Hermitian means it has a complex structure, and symmetric means really nice). There are three ways to construct this space:

(1) It is the set of positive definite 2 dimensional subspaces of $\mathbb{R}^{2,n}$

(2) It is the norm 0 vectors ω of $\mathbb{PC}^{2,n}$ with $(\omega, \bar{\omega}) = 0$.

(3) It is the vectors $x + iy \in \mathbb{R}^{1,n-1}$ with $y \in C$, where the cone C is the interior of the norm 0 cone.

Exercise 217 Show that these are the same.

15.4 Spin groups in small dimensions

Here we summarize the special properties of orthogonal and spin group is dimensions up to 8.

1. In 1 dimension the groups are discrete.

2. Here the special orthogonal and spin groups are abelian

3. The spin group $Spin_3(\mathbb{R})$ is isomorphic to the special unitary group SU_2. This is because the half-spin representation has dimension 2.

4. Spin groups have a tendency to split. The two half-spin representations have dimension 2, so we get two homomorphisms to SL_2. (Spin groups in this dimension do not always split: for example the spin group $Spin_{1,3}(\mathbb{R})$ of special relativity is locally isomorphic to $SL_2(\mathbb{C})$ which is simple modulo its center.)

5. The spin group is isomorphic to a symplectic group Sp_4. This is because the spin representation has dimension 4 and has a skew symmetric form.

6. The spin group is isomorphic to $SU_4(\mathbb{R})$. This is because the half-spin representation has dimension 4.

7. The spin group $Spin_7(\mathbb{R})$ acts transitively on the sphere S^7 in the spin representation (of dimension 8) and the subgroup fixing a point is the exceptional group G_2.

8. The two half-spin representations of dimension 8 have the same dimension as the vector representation, and have symmetric invariant bilinear forms. The spin group has a triality outer automorphism of order 3, permuting the two half-spin representations and the vector representation.

15.5 String groups

If we start with the orthogonal group (of a finite dimensional positive definite real vector space), it has nontrivial zeroth homotopy group, and we can kill this by taking its connected component, the special orthogonal group. This has non-trivial first homotopy group of order 2, and we can kill this by taking its spin double cover. This process of killing the lowest homotopy group can be continued further as

$$1 \cdots \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \mathrm{O}(n)$$

The first nonvanishing homotopy group of the spin group is the third homotopy group, which is $\mathbb{Z}$. The result of killing the third homotopy group of a spin group is called a string group. At first sight this seems impossible to construct: one problem is that any nonabelian compact Lie group has non-trivial third homotopy group. However it is possible to kill the third homotopy group if one allows non-compact infinite dimensional groups.

Stolz gave the following construction of the string group of a spin group G. Take the PU bundle P over G corresponding to a generator of $\pi_3(G)$, where PU is the projective unitary group of an infinite dimensional separable Hilbert space and is an Eilenberg-Maclane space $K(\mathbb{Z}, 2)$. This follows because the infinite dimensional unitary group U is contractible, and we have a fibration

$$1 \to \text{Circle group} \to U \to PU \to 1$$

so by the long exact sequence of homotopy groups we see that $\pi_n(S^1) = \pi_{n+1}(PU)$. Then the string group is the group of bundle automorphisms that act on the space G as left translations by elements of G. So we have an exact sequence

$$1 \to \text{Gauge group} \to \text{String group} \to \text{Spin group} \to 1$$

A similar construction works if the spin group is replaced by any simply connected simple compact Lie group, since all such groups are 2-connected and have infinite cyclic 3rd homotopy groups.

The construction of the string group from the spin group is similar to the construction of the spin group from the special orthogonal group. In the latter case one takes the $\mathbb{Z}/2\mathbb{Z}$ bundle over $SO_n(\mathbb{R})$, and the spin group is the group of bundle automorphisms lifting translations of the special orthogonal group.

In high dimensions the 4th, 5th, and 6th homotopy groups of the spin group and string group also vanish. One can continue this process further; killing the 7th homotopy group of the string group produces a group called the fivebrane group.

16 Symplectic groups

Symplectic groups are similar to orthogonal groups, but somewhat easier to handle. Over a field there are usually many different non-singular quadratic forms of given dimension, but only 1 alternating form in even dimensions, and none in odd dimensions.

Symplectic groups in dimension $2n$ are easily confused with orthogonal groups in dimension $2n+1$. They have the same dimension $n(2n+1)$, the same Weyl group, and almost the same root system, and they are locally isomorphic for $n \le 2$. In the early days of Lie group Killing thought at first that they were the same. Over fields of characteristic 2 they do become essentially the same: to see this, consider a quadratic form q in a vector space V of dimension $2n+1$. Its associated symmetric bilinear form is alternating since the characteristic is 2, so has a vector z in its kernel. So any rotation of V induces an linear transformation of the $2n$-dimensional vector space V/z preserving its induced symplectic form. So in characteristic 2 we get a homomorphism from an orthogonal group in dimension $2n+1$ to the symplectic group in dimension $2n$.

We can work out the root system in much the same way that we found the root system of an orthogonal group in even dimensions. We take the symplectic form to be the one with bolcks of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ down the diagonal. The Cartan subalgebra is then the diagonal matrices with $(\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots)$ down the diagonal, just as for orthogonal groups in even dimensions. We get elements $\pm\alpha_i \pm \alpha_j$ as roots just as for even orthogonal groups, but we also get extra roots $\pm 2\alpha_i$. This root system is called C_n. It is just like B_n except that the roots $\pm\alpha_i$ are doubled to $\pm 2\alpha_i$. In particular the Weyl group is the same in both cases and is just the group $(\mathbb{Z}/2\mathbb{Z})^n S_n$ of order $2^n n!$.

Exercise 218 Show that if n is 1 or 2 then the root system B_n is isomorphic to C_n (up to rescaling) but if $n \ge 3$ they are different.

We recall that $SP_4(\mathbb{C})$ is locally isomorphic to $SO_5(\mathbb{C})$. Form the point of view of $SO_5(\mathbb{C})$ we saw that this is related to the spin double cover of $SO_5(\mathbb{C})$, which is $Sp_4(\mathbb{C})$ as it has a 4-dimensional spin representation with an alternating form. We can see this more easily if we start from $Sp_4(\mathbb{C})$. This has a 4-dimensional representation $\mathbb{C}^4$ with an alternating form. Its alternating square has dimension 6, and has a symmetric bilinear form as the alternating 4th power is $\mathbb{C}$. However the alternating square splits as the sum of 1 and 5 dimensional pieces, as the alternating form gives an invariant 1-dimensional piece. So $Sp_4(\mathbb{C})$ has a 5-dimensional representation with a symmetric bilinear form, and therefore has a map to the orthogonal group $O_5(\mathbb{C})$.

Exercise 219 Which of the three groups $SO_5(\mathbb{R})$, $SO_{4,1}(\mathbb{R})$, $SO_{3,2}(\mathbb{R})$ is $Sp_4(\mathbb{R})$ locally isomorphic to?

There are several ways to distinguish symplectic groups in higher dimensions from orthogonal (or rather spin) groups:

- If we know that Cartan subalgebras are all conjugate, then we can distinguish orthogonal and symplectic groups by the number of long roots in their roots system: the orthogonal groups of dimension $2n+1$ have $2n(n-1)$ long roots, while symplectic groups in dimension $2n$ have $2n$ long roots, so if $n > 2$ the groups are different.
- Another way to distinguish them is to look at the dimension of the non-trivial minuscule representation (the smallest representation on which the center of the simply connected group acts non-trivially). For orthogonal groups in dimension $2n+1$ this is the spin representation of dimension $2^n = 2, 4, 8, 16, \cdots$, while for symplectic groups in dimension $2n$ it is the representation of dimension $2n = 2, 4, 6, 8, \cdots$. Again these are different if $n > 2$.

The symmetric spaces of symplectic groups are generalizations of the upper half plane called Siegel upper half planes. The Siegel upper half plane consists of matrices in $M_n(\mathbb{C})$ whose imagianry part is positive definite (so for $n = 1$ this is the usual upper half plane). We take the symplectic form given by $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ where I is the n by n identity matrix. Then the action of a symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ on an element τ of the Siegel upper half plane is given by

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)(\tau) = \frac{A\tau + B}{C\tau + D}$$

Exercise 220 Check that this is indeed a well defined action of the symplectic group on the Siegel upper half plane. (One way is to use the fact that the symplectic group is generated by matrices of the form $\left(\begin{smallmatrix} I & B \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$).

We have already seen two special cases of this before: if $n = 1$ then the symplectic group Sp_{2n} is just $SL_2(\mathbb{R})$, acting on the usual upper half plane. If $n = 2$ then $Sp_{2n}(\mathbb{R})$ is locally isomorphic to $SO_{3,2}(\mathbb{R})$, and we have seen that groups $SO_{m,2}(\mathbb{R})$ have Hermitian symmetric spaces.

The theory of modular forms on the upper half plane generalizes to Siegel modular forms on the Siegel upper half planes: for example, points of the upper half plane modulo $SL_2(\mathbb{Z}) = Sp_2(\mathbb{Z})$ correspond to complex elliptic curves, while points of the Siegel upper half plane modulo $Sp_{2n}(\mathbb{Z})$ correspond to principally polarized complex abelian varieties. This is of course all a special case of the Langlands program.