Remark 206 In terms of Galois cohomology, there an exact sequence of algebraic groups (over an algebrically closed field)

$$1 \to GL_1 \to \Gamma_V \to O_V \to 1$$

We do not necessarily get an exact sequence when taking values in some subfield.

If

$$1 \to A \to B \to C \to 1$$

is exact,

$$1 \to A(K) \to B(K) \to C(K)$$

is exact, but the map on the right need not be surjective. Instead what we get is

$$1 \to H^0(\mathrm{Gal}(\bar{K}/K), A) \to H^0(\mathrm{Gal}(\bar{K}/K), B) \to H^0(\mathrm{Gal}(\bar{K}/K), C) \to$$
$$\to H^1(\mathrm{Gal}(\bar{K}/K), A) \to \cdots$$

It turns out that $H^1(Gal(\bar{K}/K), GL_1) = 1$. However, $H^1(Gal(\bar{K}/K), \pm 1) = K^\times/(K^\times)^2$.

So from

$$1 \to GL_1 \to \Gamma_V \to O_V \to 1$$

we get

$$1 \to K^\times \to \Gamma_V(K) \to O_V(K) \to 1 = H^1(Gal(\bar{K}/K), GL_1)$$

However, taking

$$1 \to \mu_2 \to \mathrm{Spin}_V \to SO_V \to 1$$

we get

$$1 \to \pm 1 \to \mathrm{Spin}_V(K) \to SO_V(K) \xrightarrow{N} K^\times/(K^\times)^2 = H^1(\bar{K}/K, \mu_2)$$

so the non-surjectivity of N is some kind of higher Galois cohomology.

Warning 207 $\mathrm{Spin}_V \to SO_V$ is onto as a map of ALGEBRAIC GROUPS, but $\mathrm{Spin}_V(K) \to SO_V(K)$ need NOT be onto.

Example 208 Take $O_3(\mathbb{R}) \cong SO_3(\mathbb{R}) \times \{\pm 1\}$ as 3 is odd (in general $O_{2n+1}(\mathbb{R}) \cong SO_{2n+1}(\mathbb{R}) \times \{\pm 1\}$). So we have a sequence

$$1 \to \pm 1 \to \mathrm{Spin}_3(\mathbb{R}) \to SO_3(\mathbb{R}) \to 1.$$

Notice that $\mathrm{Spin}_3(\mathbb{R}) \subseteq C_3^0(\mathbb{R}) \cong \mathbb{H}$, so $\mathrm{Spin}_3(\mathbb{R}) \subseteq \mathbb{H}^\times$, and in fact we saw that it is S^3.

14.2 Covers of symmetric and alternating groups

The symmetric group on n letter can be embedded in the obvious way in $O_n(\mathbb{R})$ as permutations of coordinates. Lifting this to the pin group gives a double cover of the symmetric group, which restricts to a perfect double cover of the alternating group if n is at least 5.

Example 209 The alternating group A_5 is isomorphic to the group $PSL_2(\mathbb{F}_5)$ which has a double cover $SL_2(\mathbb{F}_5)$. Thealternating group A_6 is isomorphic to the group $PSL_2(\mathbb{F}_9)$, which has a double cover $SL_2(\mathbb{F}_9)$. (This is one way to see the extra outer automorphisms of A_6 since the group $PSL_2(\mathbb{F}_9)$ has an outer automorphism group of order 4: we can either conjugate by elements of determinant not a square, or apply a field automorphism of $\mathbb{F}_9$.)

Exercise 210 Here is another way to see the extra outer automorhisms of S_6. Show that there are 6 ways to divide the 10 edges of the complete graph on 5 points into two disjoint 5-cycles, and deduce from this that there is a homomorphism from S_5 to S_6 not conjugate to the "obvious" embedding. Then use the fact that an index n subgroup of a group gives a homomorphism to S_n to construct an outer automorphism of S_6, taking a "standard" S_5 subgroup to one of these "exceptional" ones.

In most cases this is the universal central extension of the alternating group, but there are two exceptions for $n = 6$ or 7, when the alternating group also has a perfect triple cover.

The triple cover of the alternating group A_6 was found by Valentiner and is called the Valentiner group. He found an action of A_6 on the complex projective plane, in other words a homomorphism from A_6 to $PGL_3(\mathbb{C})$, whose inverse image in the triple cover $GL_3(\mathbb{C})$ is a perfect triple cover of A_6. Here is a variation of his construction:

Exercise 211 Show that the group $PGL_3(\mathbb{F}_4)$ acts transitively on the ovals in the projective plane over F_4, where an oval is a set of 6 points such that no 3 are on a line. Show that the subgroup fixing an oval is isomorphic to A_6, acting on the 6 points of the oval. Show that the inverse image of this group in $GL_3(\mathbb{F}_4)$ is a perfect triple cover of A_6.

15 Spin groups

15.1 Spin representations of Spin and Pin groups

Notice that $\mathrm{Pin}_V(K) \subseteq C_V(K)^\times$, so any module over $C_V(K)$ gives a representation of $\mathrm{Pin}_V(K)$. We already figured out that $C_V(K)$ are direct sums of matrix algebras over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

What are the representations (modules) of complex Clifford algebras? Recall that $C_{2n}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$, which has a representations of dimension 2^n, which is called the spin representation of $\mathrm{Pin}_V(K)$ and $C_{2n+1}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C}) \times \mathbb{M}_{2^n}(\mathbb{C})$, which has 2 representations, called the spin representations of $\mathrm{Pin}_{2n+1}(K)$.

What happens if we restrict these to $\mathrm{Spin}_V(\mathbb{C}) \subseteq \mathrm{Pin}_V(\mathbb{C})$? To do that, we have to recall that $C^0_{2n}(\mathbb{C}) \cong \mathbb{M}_{2^{n-1}}(\mathbb{C}) \times \mathbb{M}_{2^{n-1}}(\mathbb{C})$ and $C^0_{2n+1}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$. So in EVEN dimensions $\mathrm{Pin}_{2n}(\mathbb{C})$ has 1 spin representation of dimension 2^n splitting into 2 HALF SPIN representations of dimension 2^{n-1} and in ODD dimensions, $\mathrm{Pin}_{2n+1}(\mathbb{C})$ has 2 spin representations of dimension 2^n which become the same on restriction to $\mathrm{Spin}_V(\mathbb{C})$.

Now we give a second description of spin representations. We will just do the even dimensional case (the odd dimensional case is similar). Suppose $\dim V = 2n$, and work over $\mathbb{C}$. Choose an orthonormal basis $\gamma_1, \dots, \gamma_{2n}$ for V, so that

$\gamma_i^2 = 1$ and $\gamma_i\gamma_j = -\gamma_j\gamma_i$. Now look at the group G generated by $\gamma_1, \dots, \gamma_{2n}$, which is finite, with order 2^{1+2n}. The representations of $C_V(\mathbb{C})$ correspond to representations of G, with -1 acting as -1 (as opposed to acting as 1). So another way to look at representations of the Clifford algebra, is to look at representations of G.

We look at the structure of G:

(1) The center is ± 1. This uses the fact that we are in even dimensions, otherwise $\gamma_1 \cdots \gamma_{2n}$ is also central.

(2) The conjugacy classes: 2 of size 1 (1 and -1), $2^{2n}-1$ of size 2 ($\pm\gamma_{i_1}\cdots\gamma_{i_n}$), so we have a total of $2^{2n}+1$ conjugacy classes, so we should have that many representations. G/center is abelian, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2n}$, which gives us 2^{2n} representations of dimension 1, so there is only one more left to find. We can figure out its dimension by recalling that the sum of the squares of the dimensions of irreducible representations gives us the order of G, which is 2^{2n+1}. So $2^{2n} \times 1^1 + 1 \times d^2 = 2^{2n+1}$, where d is the dimension of the mystery representation. Thus, $d = \pm 2^n$, so $d = 2^n$. Thus, G, and therefore $C_V(\mathbb{C})$, has an irreducible representation of dimension 2^n (as we found earlier by showing that the Clifford algebra is isomorphic to $M_{2^n}(\mathbb{C})$).

Example 212 Consider $O_{2,1}(\mathbb{R})$. As before, $O_{2,1}(\mathbb{R}) \cong SO_{2,1}(\mathbb{R}) \times (\pm 1)$, and $SO_{2,1}(\mathbb{R})$ is not connected: it has two components, separated by the spinor norm N. We have maps

$$1 \to \pm 1 \to \mathrm{Spin}_{2,1}(\mathbb{R}) \to SO_{2,1}(\mathbb{R}) \xrightarrow{N} \pm 1.$$

$\mathrm{Spin}_{2,1}(\mathbb{R}) \subseteq C^*_{2,1}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R})$, so $\mathrm{Spin}_{2,1}(\mathbb{R})$ has one 2 dimensional spin representation. So there is a map $\mathrm{Spin}_{2,1}(\mathbb{R}) \to SL_2(\mathbb{R})$; by counting dimensions and such, we can show it is an isomorphism. So $\mathrm{Spin}_{2,1}(\mathbb{R}) \cong SL_2(\mathbb{R})$.

Now let's look at some 4 dimensional orthogonal groups

Example 213 Look at $SO_4(\mathbb{R})$, which is compact. It has a complex spin representation of dimension $2^{4/2} = 4$, which splits into two half spin representations of dimension 2. We have the sequence

$$1 \to \pm 1 \to \mathrm{Spin}_4(\mathbb{R}) \to SO_4(\mathbb{R}) \to 1 \qquad (N = 1)$$

$\mathrm{Spin}_4(\mathbb{R})$ is also compact, so the image in any complex representation is contained in some unitary group. So we get two maps $\mathrm{Spin}_4(\mathbb{R}) \to SU(2) \times SU(2)$, and both sides have dimension 6 and centers of order 4. Thus, we find that $\mathrm{Spin}_4(\mathbb{R}) \cong SU(2) \times SU(2) \cong S^3 \times S^3$, which give you the two half spin representations.

So now we have done the positive definite case.

Example 214 Look at $SO_{3,1}(\mathbb{R})$. Notice that $O_{3,1}(\mathbb{R})$ has four components distinguished by the maps $\det, N \to \pm 1$. So we get

$$1 \to \pm 1 \to \mathrm{Spin}_{3,1}(\mathbb{R}) \to SO_{3,1}(\mathbb{R}) \xrightarrow{N} \pm 1 \to 1$$

We expect 2 half spin representations, which give us two homomorphisms $\mathrm{Spin}_{3,1}(\mathbb{R}) \to SL_2(\mathbb{C})$. This time, each of these homomorphisms is an isomorphism (I can't think of why right now). The $SL_2(\mathbb{C})$s are double covers of simple groups. Here, we do not get the splitting into a product as in the positive definite case. This isomorphism is heavily used in quantum field theory because $\mathrm{Spin}_{3,1}(\mathbb{R})$ is a double cover of the connected component of the Lorentz group (and $SL_2(\mathbb{C})$ is easy to work with). Note also that the center of $\mathrm{Spin}_{3,1}(\mathbb{R})$ has order 2, not 4, as for $\mathrm{Spin}_{4,0}(\mathbb{R})$. Also note that the group $PSL_2(\mathbb{C})$ acts on the compactified $\mathbb{C} \cup \{\infty\}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau) = \frac{a\tau+b}{c\tau+d}$. Subgroups of this group are called Kleinian groups. On the other hand, the group $SO_{3,1}(\mathbb{R})^+$ (identity component) acts on $\mathbb{H}^3$ (three dimensional hyperbolic space). To see this, look at the 2-sheeted hyperboloid.

One sheet of norm -1 hyperboloid is isomorphic to $\mathbb{H}^3$ under the induced metric. In fact, we'll define hyperbolic space that way. Topologists are very interested in hyperbolic 3-manifolds, which are $\mathbb{H}^3/(\text{discrete subgroup of } SO_{3,1}(\mathbb{R}))$. If you use the fact that $SO_{3,1}(\mathbb{R}) \cong PSL_2(\mathbb{R})$, then you see that these discrete subgroups are in fact Kleinian groups.

There are lots of exceptional isomorphisms in small dimension, all of which are very interesting, and almost all of them can be explained by spin groups.

Example 215 $O_{2,2}(\mathbb{R})$ has 4 components (given by det, N); $C^0_{2,2}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R})$, which induces an isomorphism $\mathrm{Spin}_{2,2}(\mathbb{R}) \to SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, which gives the two half spin representations. Both sides have dimension 6 with centers of order 4. So this time we get two non-compact groups. Let us look at the fundamental group of $SL_2(\mathbb{R})$, which is $\mathbb{Z}$, so the fundamental group of $\mathrm{Spin}_{2,2}(\mathbb{R})$ is $\mathbb{Z} \oplus \mathbb{Z}$. As we recall, $\mathrm{Spin}_{4,0}(\mathbb{R})$ and $\mathrm{Spin}_{3,1}(\mathbb{R})$ were both simply connected. This shows that SPIN GROUPS NEED NOT BE SIMPLY CONNECTED. So we can take covers of it. What do the corresponding covers (e.g. the universal cover) of $\mathrm{Spin}_{2,2}(\mathbb{R})$ look like? This is hard to describe because for finite dimensional complex representations, we get finite dimensional representations of the Lie algebra L, which correspond to the finite dimensional representations of $L \otimes \mathbb{C}$, which correspond to the finite dimensional representations of $L' =$ Lie algebra of $\mathrm{Spin}_{4,0}(\mathbb{R})$, which correspond to the finite dimensional representations of $\mathrm{Spin}_{4,0}(\mathbb{R})$, which has no covers because it is simply connected. This means that any finite dimensional representation of a cover of $\mathrm{Spin}_{2,2}(\mathbb{R})$ actually factors through $\mathrm{Spin}_{2,2}(\mathbb{R})$. So there is no way to describe these things with finite matrices, and infinite dimensional representations are hard.

To summarize, the ALGEBRAIC GROUP $\mathrm{Spin}_{2,2}$ is simply connected (as an algebraic group) (think of an algebraic group as a functor from rings to groups), which means that it has no algebraic central extensions. However, the LIE GROUP $\mathrm{Spin}_{2,2}(\mathbb{R})$ is NOT simply connected; it has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. This problem does not happen for COMPACT Lie groups (where every finite cover is algebraic).

We have done $O_{4,0}, O_{3,1}$, and $O_{2,2}$, from which we can obviously get $O_{1,3}$ and $O_{0,4}$. Note that $O_{4,0}(\mathbb{R}) \cong O_{0,4}(\mathbb{R})$, $SO_{4,0}(\mathbb{R}) \cong SO_{0,4}(\mathbb{R})$, $\mathrm{Spin}_{4,0}(\mathbb{R}) \cong \mathrm{Spin}_{0,4}(\mathbb{R})$. However, $\mathrm{Pin}_{4,0}(\mathbb{R}) \ncong \mathrm{Pin}_{0,4}(\mathbb{R})$. These two are hard to distinguish. We have

Take a reflection (of order 2) in $O_{4,0}(\mathbb{R})$, and lift it to the Pin groups. What is the order of the lift? The reflection vector v, with $v^2 = \pm 1$ lifts to the element $v \in \Gamma_V(\mathbb{R}) \subseteq C_V^*(\mathbb{R})$. Notice that $v^2 = 1$ in the case of $\mathbb{R}^{4,0}$ and $v^2 = -1$ in the case of $\mathbb{R}^{0,4}$, so in $\mathrm{Pin}_{4,0}(\mathbb{R})$, the reflection lifts to something of order 2, but in $\mathrm{Pin}_{0,4}(\mathbb{R})$, you get an element of order 4!. So these two groups are different.

Two groups are *isoclinic* if they are confusingly similar. A similar phenomenon is common for groups of the form $2 \cdot G \cdot 2$, which means it has a center of order 2, then some group G, and the abelianization has order 2. Watch out.

Exercise 216 $\mathrm{Spin}_{3,3}(\mathbb{R}) \cong SL_4(\mathbb{R})$.

15.2 Triality

This is a special property of 8 dimensional orthogonal groups. Recall that $O_8(\mathbb{C})$ has the root system D_4, which has an extra symmetry of order three.

But $O_8(\mathbb{C})$ and $SO_8(\mathbb{C})$ do NOT have corresponding symmetries of order three. The thing that does have the symmetry of order three is the spin group. The group $\mathrm{Spin}_8(\mathbb{R})$ DOES have "extra" order three symmetry. We can see it as follows. Look at the half spin representations of $\mathrm{Spin}_8(\mathbb{R})$. Since this is a spin group in even dimension, there are two. $C_{8,0}(\mathbb{R}) \cong \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \times \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \cong \mathbb{M}_8(\mathbb{R}) \times \mathbb{M}_8(\mathbb{R})$. So $\mathrm{Spin}_8(\mathbb{R})$ has two 8 dimensional real half spin representations. But the spin group is compact, so it preserves some quadratic form, so you get 2 homomorphisms $\mathrm{Spin}_8(\mathbb{R}) \to SO_8(\mathbb{R})$. So $\mathrm{Spin}_8(\mathbb{R})$ has THREE 8 dimensional representations: the half spins, and the one from the map to $SO_8(\mathbb{R})$. These maps $\mathrm{Spin}_8(\mathbb{R}) \to SO_8(\mathbb{R})$ lift to Triality automorphisms $\mathrm{Spin}_8(\mathbb{R}) \to \mathrm{Spin}_8(\mathbb{R})$. The center of $\mathrm{Spin}_8(\mathbb{R})$ is $(\mathbb{Z}/2) + (\mathbb{Z}/2)$ because the center of the Clifford group is $\pm 1, \pm \gamma_1 \cdots \gamma_8$. There are 3 non-trivial elements of the center, and quotienting by any of these gives you something isomorphic to $SO_8(\mathbb{R})$. This is special to 8 dimensions.