14.1 Clifford groups, Spin groups, and Pin groups

In this section, we define Clifford groups of a vector space V with a quadratic form over a field K, denoted $\Gamma_V(K)$. They are central extensions of orthogonal groups that fit into an exact sequence

$$1 \to K^\times \to \Gamma_V(K) \to O_V(K) \to 1.$$

Definition 195 *We define α to be the automorphism of $C_V(K)$ induced by -1 on V (in other words the automorphism which acts by -1 on odd elements and 1 on even elements)). The Clifford group $\Gamma_V(K)$ is the group of homogeneous invertible elements $x \in C_V(K)$ such that $xV\alpha(x)^{-1} \subseteq V$ (recall that $V \subseteq C_V(K)$). This also gives an action of $\Gamma_V(K)$ on V.*

Many authors leave out the α, which is a mistake, though not a serious one, and use xVx^{-1} instead of $xV\alpha(x)^{-1}$. Our definition (introduced by Atiyah, Bott, and Schapiro) is better for the following reasons:

1. It is the correct superalgebra sign. The superalgebra convention says that whenever you exchange two elements of odd degree, you pick up a minus sign, and V is odd.

2. Putting α in makes the theory much cleaner in odd dimensions. For example, we will see that the described action gives a map $\Gamma_V(K) \to O_V(K)$ which is onto if we use α, but not if we do not. (You get $SO_V(K)$ rather than $O_V(K)$ in odd dimensions without the α, which is not a disaster, but is annoying.)

Lemma 196 *The elements of $\Gamma_V(K)$ that act trivially on V are the elements of $K^\times \subseteq \Gamma_V(K) \subseteq C_V(K)$.*

Proof Suppose $a_0 + a_1 \in \Gamma_V(K)$ acts trivially on V, with a_0 even and a_1 odd. Then $(a_0 + a_1)v = v\alpha(a_0 + a_1) = v(a_0 - a_1)$. Matching up even and odd parts, we get $a_0 v = va_0$ and $a_1 v = -va_1$. Choose an orthogonal basis $\gamma_1, \dots, \gamma_n$ for V. (All these results are true in characteristic 2, but you have to work harder: you cannot go around choosing orthogonal bases because they may not exist.) We may write

$$a_0 = x + \gamma_1 y$$

where $x \in C_V^0(K)$ and $y \in C_V^1(K)$ and neither x nor y contain a factor of γ_1, so $\gamma_1 x = x\gamma_1$ and $\gamma_1 y = y\gamma_1$. Applying the relation $a_0 v = va_0$ with $v = \gamma_1$, we see that $y = 0$, so a_0 contains no monomials with a factor γ_1.

Repeat this procedure with v equal to the other basis elements to show that $a_0 \in K^\times$ (since it cannot have any γ's in it). Similarly, write $a_1 = y + \gamma_1 x$, with x and y not containing a factor of γ_1. Then the relation $a_1\gamma_1 = -\gamma_1 a_1$ implies that $x = 0$. Repeating with the other basis vectors, we conclude that $a_1 = 0$.

So $a_0 + a_1 = a_0 \in K \cap \Gamma_V(K) = K^\times$. □ Now we define $-^T$ to be the identity on V, and extend it to an anti-automorphism of $C_V(K)$ ("anti" means that $(ab)^T = b^T a^T$). Do not confuse $a \mapsto \alpha(a)$ (automorphism), $a \mapsto a^T$ (anti-automorphism), and $a \mapsto \alpha(a^T)$ (anti-automorphism).

Now we define the *spinor norm* of $a \in C_V(K)$ by $N(a) = aa^T$. We also define a twisted version: $N^\alpha(a) = a\alpha(a)^T$.

Proposition 197

1. *The restriction of N to $\Gamma_V(K)$ is a homomorphism whose image lies in $K^\times$. (N is a mess on the rest of $C_V(K)$.)*
2. *The action of $\Gamma_V(K)$ on V is orthogonal. That is, we have a homomorphism $\Gamma_V(K) \to O_V(K)$.*

Proof First we show that if $a \in \Gamma_V(K)$, then $N^\alpha(a)$ acts trivially on V.

$$N^\alpha(a)\, v\, \alpha\big(N^\alpha(a)\big)^{-1} = a\alpha(a)^T v\Big(\alpha(a)\, \underbrace{\alpha\big(\alpha(a)^T\big)}_{=a^T}\Big)^{-1} \tag{5}$$

$$= a\, \underbrace{\alpha(a)^T v (a^{-1})^T}_{=(a^{-1}v^T\alpha(a))^T}\, \alpha(a)^{-1} \tag{6}$$

$$= aa^{-1} v\alpha(a)\alpha(a)^{-1} \tag{7}$$

$$= v \tag{8}$$

So by Lemma ???, $N^\alpha(a) \in K^\times$. This implies that N^α is a homomorphism on $\Gamma_V(K)$ because

$$N^\alpha(a)N^\alpha(b) = a\alpha(a)^T N^\alpha(b) \tag{9}$$

$$= aN^\alpha(b)\alpha(a)^T \qquad (N^\alpha(b) \text{ is central}) \tag{10}$$

$$= ab\alpha(b)^T\alpha(a)^T \tag{11}$$

$$= (ab)\alpha(ab)^T = N^\alpha(ab). \tag{12}$$

After all this work with N^α, what we're really interested is N. On the even elements of $\Gamma_V(K)$, N agrees with N^α, and on the odd elements, $N = -N^\alpha$. Since $\Gamma_V(K)$ consists of homogeneous elements, N is also a homomorphism from $\Gamma_V(K)$ to $K^\times$. This proves the first statement of the proposition.

Finally, since N is a homomorphism on $\Gamma_V(K)$, the action on V preserves the quadratic form N of V. Thus, we have a homomorphism $\Gamma_V(K) \to O_V(K)$. $\square$

On V, N coincides with the quadratic form N. Some authors seem not to have noticed this, and use different letters for the norm N and the spinor norm N on V. Sometimes they use a porrly chosen sign convention which makes them different.

Now we analyze the homomorphism $\Gamma_V(K) \to O_V(K)$. Lemma ??? says exactly that the kernel is $K^\times$. Next we will show that the image is all of $O_V(K)$. Say $r \in V$ and $N(r) \neq 0$.

$$rv\alpha(r)^{-1} = -rv\frac{r}{N(r)} = v - \frac{vr^2 + rvr}{N(r)}$$

$$= v - \frac{(v,r)}{N(r)}r \tag{13}$$

$$= \begin{cases} -r & \text{if } v = r \\ v & \text{if } (v,r) = 0 \end{cases} \tag{14}$$

Thus, r is in $\Gamma_V(K)$, and it acts on V by reflection through the hyperplane $r^\perp$. One might deduce that the homomorphism $\Gamma_V(K) \to O_V(K)$ is surjective because $O_V(K)$ is generated by reflections. This is wrong; $O_V(K)$ is *not* always generated by reflections!

Exercise 198 Let $H = \mathbb{F}_2^2$, with the quadratic form $x^2 + y^2 + xy$, and let $V = H \oplus H$. Prove that $O_V(\mathbb{F}_2)$ is not generated by reflections.

Solution 199 *In H, the norm of any non-zero vector is 1. It is immediate to check that the reflection of a non-zero vector v through another non-zero vector u is*

$$r_u(v) = \begin{cases} u & \text{if } u = v \\ v + u & \text{if } u \neq v \end{cases}$$

so reflection through a non-zero vector fixes that vector and swaps the two other non-zero vectors. Thus, the reflection in H generate the symmetric group on three elements S_3, acting on the three non-zero vectors.

If u and v are non-zero vectors, then $(u, v) \in H \oplus H$ has norm $1 + 1 = 0$, so one cannot reflect through it. Thus, every reflection in V is "in one of the H's," so the group generated by reflections is $S_3 \times S_3$. However, swapping the two H's is clearly an orthogonal transformation, so reflections do not generate $O_V(\mathbb{F}_2)$.

Remark 200 This is the only counterexample. For any other vector space and any other non-degenerate quadratic form on this space, $O_V(K)$ is generated by reflections. The map $\Gamma_V(K) \to O_V(K)$ is surjective even in the example above. Also, in every case except the example above, $\Gamma_V(K)$ is generated as a group by non-zero elements of V (i.e. every element of $\Gamma_V(K)$ is a monomial).

Remark 201 Equation ??? is the definition of the reflection of v through r. It is only possible to reflect through vectors of non-zero norm. Reflections in characteristic 2 are strange; strange enough that people don't call them reflections, they call them *transvections.*

Definition 202

$$Pin_V(K) = \{x \in \Gamma_V(K) | N(x) = 1\}$$

, and

$$Spin_V(K) = Pin_V^0(K)$$

, the even elements of $Pin_V(K)$.

On $K^\times$, the spinor norm is given by $x \mapsto x^2$, so the elements of spinor norm 1 are $= \pm 1$.

$$\begin{pmatrix} 1 & \to & \pm 1 & \to & Pin_V(k) & \to & \Omega_V & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & k^* & \to & \Gamma_V(k) & \to & O_V(k) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & k^{*2} & \to & k^* & \to & k^*/k^{*2} & \to & 1 \end{pmatrix}$$

where the rows are exact, $K^\times$ is in the center of $\Gamma_V(K)$ (this is obvious, since $K^\times$ is in the center of $C_V(K)$), and $N : O_V(K) \to K^\times/(K^\times)^2$ is the unique homomorphism sending reflection through $r^\perp$ to $N(r)$ modulo $(K^\times)^2$.

To see exactness of the top sequence, note that the kernel of ϕ is $K^\times \cap \mathrm{Pin}_V(K) = \pm 1$, and that the image of $\mathrm{Pin}_V(K)$ in $O_V(K)$ is exactly the elements of norm 1. The bottom sequence is similar, except that the image of $\mathrm{Spin}_V(K)$ is not all of $O_V(K)$, it is only $SO_V(K)$; by Remark **??**, every element of $\Gamma_V(K)$ is a product of elements of V, so every element of $\mathrm{Spin}_V(K)$ is a product of an even number of elements of V. Thus, its image is a product of an even number of reflections, so it is in $SO_V(K)$.

These maps are NOT always onto, but there are many important cases when they are, such as when V has a positive definite quadratic form. The image is the set of elements of $O_V(K)$ or $SO_V(K)$ that have spinor norm 1 in $K^\times/(K^\times)^2$.

What is $N : O_V(K) \to K^\times/(K^\times)^2$? It is the UNIQUE homomorphism such that $N(a) = N(r)$ if a is reflection in $r^\perp$, and r is a vector of norm $N(r)$.

Example 203 Take V to be a positive definite vector space over $\mathbb{R}$. Then N maps to 1 in $\mathbb{R}^\times/(\mathbb{R}^\times)^2 = \pm 1$ (because N is positive definite). So the spinor norm on $O_V(\mathbb{R})$ is trivial.

So if V is positive definite, we get double covers

$$1 \to \pm 1 \to \mathrm{Pin}_V(\mathbb{R}) \to O_V(\mathbb{R}) \to 1$$

$$1 \to \pm 1 \to \mathrm{Spin}_V(\mathbb{R}) \to SO_V(\mathbb{R}) \to 1$$

This will account for the weird double covers we saw before.

What if V is negative definite? Every reflection now has image -1 in $\mathbb{R}^\times/(\mathbb{R}^\times)^2$, so the spinor norm N is the same as the determinant map $O_V(\mathbb{R}) \to \pm 1$.

So in order to find interesting examples of the spinor norm, you have to look at cases that are neither positive definite nor negative definite.

Let's look at Lorentz space: $\mathbb{R}^{1,3}$.

Reflection through a vector of norm < 0 (spacelike vector, P: parity reversal) has spinor norm -1, det -1 and reflection through a vector of norm > 0 (timelike vector, T: time reversal) has spinor norm $+1$, det -1. So $O_{1,3}(\mathbb{R})$ has 4 components (it is not hard to check that these are all the components), usually called 1, P, T, and PT.

Example 204 The Weyl group of F_4 is generated by reflections of vectors of norms 1 and 2. It is a subgroup of $O_4(\mathbb{Q})$ so the spinor norm is a homomorphism to Q^*/Q^{*2}. So by combinign this with the determinant map, we get a homomorphism of this Weyl group onto the Klein 4-group $(\mathbb{Z}/2\mathbb{Z})^2$, mapping reflections of norm 1 or norm 2 vectors onto two different non-trivial elements. Similarly we see immediately that the Weyl group of B_n has a homomorphism onto the Klein 4-group.

Example 205 The groups $PSO_n(\mathbb{R})$ are simple for $n \geq 5$, so one might guess by analogy that the groups $PSO_n(\mathbb{Q})$ are also simple, but the spinor norm shows immediately that they are not. In fact the spinor norm maps $O_n(\mathbb{Q})$ onto the infinite index 2 subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ represented by positive elements, so the abelianization of $PSO_n(\mathbb{Q})$ is infinite.