If $\{v_1, \ldots, v_n\}$ is a basis for V, then $\{v_{i_1} \cdots v_{i_k} | i_1 < \cdots < i_k,\ k \leq n\}$ spans $C_V(K)$, so the dimension of $C_V(K)$ is less than or equal to $2^{\dim V}$. As usual with objects given by generators and relations, the harder problem is showing that it cannot be smaller.

Now let's try to analyze larger Clifford algebras more systematically. What is $C_{U \oplus V}$ in terms of C_U and C_V? One might guess $C_{U\oplus V} \cong C_U \otimes C_V$. For the usual definition of tensor product, this is false (e.g. $C_{1,1}(\mathbb{R}) \neq C_{1,0}(\mathbb{R}) \otimes C_{0,1}(\mathbb{R})$). However, for the *superalgebra* definition of tensor product, this is correct. The superalgebra tensor product is the regular tensor product of vector spaces, with the product given by $(a \otimes b)(c \otimes d) = (-1)^{\deg b \cdot \deg c} ac \otimes bd$ for homogeneous elements a, b, c, and d. For the moment we will forget about superalgebras, and naively calculate with the ordinary tensor product.

Let's specialize to the case $K = \mathbb{R}$ and try to compute $C_{U \oplus V}(K)$. Assume for the moment that $\dim U = m$ is even. Take $\alpha_1, \ldots, \alpha_m$ to be an orthogonal basis for U and let $\beta_1, \ldots, \beta_n$ to be an orthogonal basis for V. Then set $\gamma_i = \alpha_1 \alpha_2 \cdots \alpha_m \beta_i$. What are the relations between the α_i and the γ_j? We have

$$\alpha_i \gamma_j = \alpha_i \alpha_1 \alpha_2 \cdots \alpha_m \beta_j = \alpha_1 \alpha_2 \cdots \alpha_m \beta_i \alpha_i = \gamma_j \alpha_i$$

since $\dim U$ is even, and α_i anti-commutes with everything except itself.

$$\begin{aligned}
\gamma_i \gamma_j &= \gamma_i \alpha_1 \cdots \alpha_m \beta_j \\
&= \alpha_1 \cdots \alpha_m \gamma_i \beta_j \\
&= \alpha_1 \cdots \alpha_m \alpha_1 \cdots \alpha_m \underbrace{\beta_i \beta_j}_{-\beta_j \beta_i} \\
&= -\gamma_j \gamma_i \\
\gamma_i^2 &= \alpha_1 \cdots \alpha_m \alpha_1 \cdots \alpha_m \beta_i \beta_i \\
&= (-1)^{\frac{m(m-1)}{2}} \alpha_1^2 \cdots \alpha_m^2 \beta_i^2 \\
&= (-1)^{m/2} \alpha_1^2 \cdots \alpha_m^2 \beta_i^2 \qquad\qquad (m \text{ even})
\end{aligned}$$

So the γ_i's commute with the α_i and satisfy the relations of some Clifford algebra. Thus, we've shown that $C_{U\oplus V}(K) \cong C_U(K) \otimes C_W(K)$, where W is V with the quadratic form multiplied by

$$(-1)^{\frac{1}{2}\dim U} \alpha_1^2 \cdots \alpha_m^2 = (-1)^{\frac{1}{2}\dim U} \cdot \text{discriminant}(U),$$

and this is the usual tensor product of algebras over $\mathbb{R}$.

Taking $\dim U = 2$, we find that

$$\begin{aligned}
C_{m+2,n}(\mathbb{R}) &\cong \mathbb{M}_2(\mathbb{R}) \otimes C_{n,m}(\mathbb{R}) \\
C_{m+1,n+1}(\mathbb{R}) &\cong \mathbb{M}_2(\mathbb{R}) \otimes C_{m,n}(\mathbb{R}) \\
C_{m,n+2}(\mathbb{R}) &\cong \mathbb{H} \otimes C_{n,m}(\mathbb{R})
\end{aligned}$$

where the indices switch whenever the discriminant is positive. Using these formulas, we can reduce any Clifford algebra to tensor products of things like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{M}_2(\mathbb{R})$.

Recall the rules for taking tensor products of matrix algebras (all tensor products are over $\mathbb{R}$).

- $\mathbb{R} \otimes X \cong X$.
- $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{M}_2(\mathbb{C})$.
  This follows from the isomorphism $\mathbb{C} \otimes C_{m,n}(\mathbb{R}) \cong C_{m+n}(\mathbb{C})$.
- $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.
- $\mathbb{H} \otimes \mathbb{H} \cong \mathbb{M}_4(\mathbb{R})$.
  This follows by thinking of the action on $\mathbb{H} \cong \mathbb{R}^4$ given by $(x \otimes y) \cdot z = xz\overline{y}$.
- $\mathbb{M}_m\big(\mathbb{M}_n(X)\big) \cong \mathbb{M}_{mn}(X)$.
- $\mathbb{M}_m(X) \otimes \mathbb{M}_n(Y) \cong \mathbb{M}_{mn}(X \otimes Y)$.

Filling in the middle of the table is easy because you can move diagonally by tensoring with $\mathbb{M}_2(\mathbb{R})$. It is easy to see that $C_{8+m,n}(\mathbb{R}) \cong C_{m,n+8}(\mathbb{R}) \cong C_{m,n} \otimes \mathbb{M}_{16}(\mathbb{R})$, which gives the table a kind of mod 8 periodicity. There is a more precise way to state this: $C_{m,n}(\mathbb{R})$ and $C_{m',n'}(\mathbb{R})$ are *super Morita equivalent* if and only if $m - n \equiv m' - n' \mod 8$.

We found that the structure of a Clifford algebra depends heavily on $m - n$ mod 8. The explanation of this was not all that satisfactory as it seemed to be a fluke coming at the end of a long calculation. There is a hidden cyclic group of order 8 controlling this, given by the super Brauer group of the reals. The usual Brauer group of a field consists of the finite dimensional central division rings, with the group product given by taking tensor products (modulo taking matrix rings). For example, the Brauer group of the reals has order 2, with elements the reals and the quaternions, and the Brauer group of the complex numbers has order 1. The super Brauer group is defined similarly except we use super divison algebras: this means every nonzero HOMOGENEOUS element is invertible. The 8 elements are represented by the reals, the quaternions, and the algebras $\mathbb{R}[\epsilon]$, $\mathbb{C}[\epsilon]$, $\mathbb{H}[\epsilon]$, where ϵ is odd, $\epsilon^2 = \pm 1$ and $x\epsilon = \epsilon\overline{x}$ for x in the even part.

Exercise 192 Work out how the super division algebras over $\mathbb{R}$ correspond to elements of a cyclic group of order 8 up to super Morita equivalence, under the super tensor product. Find the 8 algebras underlying them if one forgets the grading and compare these with Clifford algebras.

This mod 8 periodicity turns up in several other places:

1. Real Clifford algebras $C_{m,n}(\mathbb{R})$ and $C_{m',n'}(\mathbb{R})$ are super Morita equivalent if and only if $m - n \equiv m' - n' \mod 8$.
2. *Bott periodicity*, which says that stable homotopy groups of orthogonal groups are periodic mod 8.
3. Real K-theory is periodic with a period of 8.
4. Even unimodular lattices (such as the E_8 lattice) exist in $\mathbb{R}^{m,n}$ if and only if $m - n \equiv 0 \mod 8$. More generally even integral lattices tend to have a strong period 8 behavior: for example $\sum_{\lambda \in L'/L} e^{\pi i \lambda^2} = e^{2\pi i \text{signature}/8} \sqrt{|\text{discriminant}|}$. For 1-dimensional lattices this is more or less Gauss's law of quadratic reciprocity in terms of Gauss sums.

5. The Super Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8\mathbb{Z}$.

Recall that $C_V(\mathbb{R}) = C_V^0(\mathbb{R}) \oplus C_V^1(\mathbb{R})$, where $C_V^1(\mathbb{R})$ is the odd part and $C_V^0(\mathbb{R})$ is the even part. We will need to know the structure of $C_{m,n}^0(\mathbb{R})$, which controls special orthogonal groups in the same way that Clifford algebras control orthogonal groups. Fortunately, this is easy to compute in terms of smaller Clifford algebras. Let $\dim U = 1$, with γ a basis for U and let $\gamma_1, \ldots, \gamma_n$ an orthogonal basis for V. Then $C_{U\oplus V}^0(K)$ is generated by $\gamma\gamma_1, \ldots, \gamma\gamma_n$. We compute the relations

$$\gamma\gamma_i \cdot \gamma\gamma_j = -\gamma\gamma_j \cdot \gamma\gamma_i$$

for $i \neq j$, and

$$(\gamma\gamma_i)^2 = (-\gamma^2)\gamma_i^2$$

So $C_{U\oplus V}^0(K)$ is itself the Clifford algebra $C_W(K)$, where W is V with the quadratic form multiplied by $-\gamma^2 = -\mathrm{disc}(U)$. Over $\mathbb{R}$, this tells us that

$$\begin{aligned} C_{m+1,n}^0(\mathbb{R}) &\cong C_{n,m}(\mathbb{R}) && \text{(mind the indices)} \\ C_{m,n+1}^0(\mathbb{R}) &\cong C_{m,n}(\mathbb{R}). \end{aligned}$$

Remark 193 For complex Clifford algebras, the situation is similar, but easier. One finds that $C_{2m}(\mathbb{C}) \cong \mathbb{M}_{2^m}(\mathbb{C})$ and $C_{2m+1}(\mathbb{C}) \cong \mathbb{M}_{2^m}(\mathbb{C}) \oplus \mathbb{M}_{2^m}(\mathbb{C})$, with $C_n^0(\mathbb{C}) \cong C_{n-1}(\mathbb{C})$. You could figure these out by tensoring the real algebras with $\mathbb{C}$ if you wanted. We see a mod 2 periodicity now. Bott periodicity for the unitary group is mod 2.

Exercise 194 Find the non-trivial finite dimensional super division algebra with center $\mathbb{C}$.

Clifford algebras are analogous to the algebra of differential operators, which are given by generators x_i and D_i with relations tht the x_i commute with each other, the D_i commute with each other, and $D_i x_j - x_j D_i = 1$. If we put a skew symmetric form on the vector space spanned by the x_i and D_i so that $\langle x_i, x_j\rangle = 0$, $\langle D_i, D_j\rangle = 0$, $\langle D_i, x_j\rangle = 1$ if $i = j$ and 0 otherwise, then the algebra of differential operators is generated by this space with the relations $ab - ba = \langle a, b\rangle$. This is similar to Clifford algebras which (in characteristic not 2) have relations $ab + ba = (a, b)$ for a symmetric form. If you work with super vector spaces, then these two constructions become special cases of the same construction.

Clifford algebras can also be obtained as a quotient of a Heisenberg superalgebra, in the same way that the algebra of differential operators is a qotient of a Heisenberg algebra. So the study of Clifford algebras and their representations is essentially the study of the Heisenberg superalgebra. This again demonstrates that it is really more natural to work with super vector spaces rather than vector spaces when studying Clifford algebras, but we will mostly just use vector spaces and just point out the changes needed for using superspaces.