In fact we can extend the root system $D_4 \subset B_4$ to a bigger root system called F_4 by adding in these 16 norm 1 roots. We will see later that this is the root system of an exceptional Lie group. We have essentially worked out the Weyl group of F_4: it is the same as the automorphism group of the root system D_4 and so has order $2^3.4!.6 = 2^7.3^2$.

There is another way to extend D_m to a bigger root system. If we put $m = 8$ then the vectors $(\pm 1/2, \pm 1/2, ...,)$ happen to have norm 2, and all norm 2 vectors of an integral lattice are roots (meaning that their reflections act on the lattice). So if we add one of these cosets to D_8 we get an integral lattice E_8 with $112 + 2^7 = 240$ vectors of norm 2. In other words the roots are the vectors $(\cdots, \pm 1, \cdots, \pm 1, \cdots)$ with two non-zero entries and the vectors $(\pm 1/2, \pm 1/2, ...,)$ with even sum. This root system also corresponds to an exceptional Lie algebra of dimension $8 + 240 = 248$.

Exercise 182 Define the root systems E_7 and E_6 to be the roots of E_8 whose first 2 or 3 coordinates are equal. Show that E_7 has 126 roots and E_6 has 72 roots. (These will turn out to correspond to Lie algebras of dimensions $126 + 7 = 133$ and $72 + 6 = 78$.)

The lattice E_8 is the smallest example of a unimodular integral positive definite lattice that is not a sum of copies of $\mathbb{Z}$. We can form similar unimodular lattices D_m^+ from D_m whenever m is divisible by 4; they are even lattices if m is divisible by 8. However if $m > 8$ they have the same roots as D_m so do not give new Lie algebras or root systems.

The lattice for $m = 16$ was used by Milnor to give a negative answer the question "can you hear the shape of a drum", in other words is a compact Riemannian manifold determined by the spectrum of its Laplacian. The spectrum of a toroidal drum is given by the number of vectors of various norms of the corresponding lattice. The lattices $E_8 + E_8$ and D_{16}^+ are distinct, and the theory of modular forms shows that they have the same theta function, in other words the same number of vectors of every norm, so the corresponding tori are not isomorphic but have the same spectrums.

Exercise 183 Show that the root systems $E_8 + E_8$ and D_{16} in $\mathbb{R}^{16}$ have the same number of roots but are not isomorphic.

14 Clifford algebras

With Lie algebras of small dimensions, we have seen that there are numerous accidental isomorphisms. Almost all of these can be explained with Clifford algebras and Spin groups.

Motivational examples that we'd like to explain:

1. $SO_2(\mathbb{R}) = S^1$: S^1 can double cover S^1 itself.
2. $SO_3(\mathbb{R})$: has a simply connected double cover S^3.
3. $SO_4(\mathbb{R})$: has a simply connected double cover $S^3 \times S^3$.
4. $SO_5(\mathbb{C})$: Look at $Sp_4(\mathbb{C})$, which acts on $\mathbb{C}^4$ and on $\Lambda^2(\mathbb{C}^4)$, which is 6 dimensional, and decomposes as $5 \oplus 1$. $\Lambda^2(\mathbb{C}^4)$ has a symmetric bilinear

form given by $\Lambda^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{C}^4) \to \Lambda^4(\mathbb{C}^4) \simeq \mathbb{C}$, and $Sp_4(\mathbb{C})$ preserves this form. You get that $Sp_4(\mathbb{C})$ acts on $\mathbb{C}^5$, preserving a symmetric bilinear form, so it maps to $SO_5(\mathbb{C})$. You can check that the kernel is ± 1. So $Sp_4(\mathbb{C})$ is a double cover of $SO_5(\mathbb{C})$.

5. $SO_5(\mathbb{C})$: $SL_4(\mathbb{C})$ acts on $\mathbb{C}^4$, and we still have our 6 dimensional $\Lambda^2(\mathbb{C}^4)$, with a symmetric bilinear form. So you get a homomorphism $SL_4(\mathbb{C}) \to SO_6(\mathbb{C})$, which you can check is surjective, with kernel ± 1.

So we have double covers S^1, S^3, $S^3 \times S^3$, $Sp_4(\mathbb{C})$, $SL_4(\mathbb{C})$ of the orthogonal groups in dimensions 2,3,4,5, and 6, respectively. All of these look completely unrelated. We will give a uniform construction of double covers of all orthogonal groups using Clifford algebras.

Example 184 We have not yet defined Clifford algebras, but to motivate the definition here are some examples of Clifford algebras over $\mathbb{R}$.

- $\mathbb{C}$ is generated by $\mathbb{R}$, together with i, with $i^2 = -1$
- $\mathbb{H}$ is generated by $\mathbb{R}$, together with i, j, each squaring to -1, with $ij + ji = 0$.
- Dirac wanted a square root for the operator $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$ (the wave operator in 4 dimensions). He supposed that the square root is of the form $A = \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} + \gamma_4 \frac{\partial}{\partial t}$ and compared coefficients in the equation $A^2 = \nabla$. Doing this yields $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1$, $\gamma_4^2 = -1$, and $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$ for $i \neq j$.

  Dirac solved this by taking the γ_i to be 4×4 complex matrices. A operates on vector-valued functions on space-time.

Definition 185 *A* Clifford algebra *over $\mathbb{R}$ is generated by elements $\gamma_1, \ldots, \gamma_n$ such that $\gamma_i^2 = \pm 1$, and $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$ for $i \neq j$.*

This is a rather clumsy and ad hoc definition. Let's try again:

Definition 186 (better definition) *Suppose V is a vector space over a field K, with some quadratic form $N : V \to K$. (N is a* quadratic form *if it is a homogeneous polynomial of degree 2 in the coefficients with respect to some basis.) Then the* Clifford algebra *$C_V(K)$ is the associative algebra generated by the vector space V, with relations $v^2 = N(v)$.*

Of course this definition also works for quadratic forms on modules over rings, or sheaves over a space, and so on, and much of the basic theory of Clifford algebras can be extended to these cases.

We know that $N(\lambda v) = \lambda^2 N(v)$ and that the expression $(a, b) := N(a + b) - N(a) - N(b)$ is bilinear. If the characteristic of K is not 2, we have $N(a) = \frac{(a,a)}{2}$. Thus, we can work with symmetric bilinear forms instead of quadratic forms so long as the characteristic of K is not 2. We will use quadratic forms so that everything works in characteristic 2. (Characteristic 2 is notoriously tricky for bilinear and quadratic forms and we will not be working in characteristic 2, but if we can pick up this case for free just by using the right definition we may as well.)

Warning 187 Some authors (mainly in index theory) use the opposite sign convention $v^2 = -N(v)$. This is a convention introduced by Atiyah and Bott.

Some people add a factor of 2 somewhare, which usually does not matter, but is wrong in characteristic 2.

Example 188 Take $V = \mathbb{R}^2$ with basis i, j, and with $N(xi + yj) = -x^2 - y^2$. Then the relations are $(xi + yj)^2 = -x^2 - y^2$ are exactly the relations for the quaternions: $i^2 = j^2 = -1$ and $(i + j)^2 = i^2 + ij + ji + j^2 = -2$, so $ij + ji = 0$.

Remark 189 If the characteristic of K is not 2, a "completing the square" argument shows that any quadratic form is isomorphic to $c_1 x_1^2 + \cdots + c_n x_n^2$, and if one can be obtained from another other by permuting the c_i and multiplying each c_i by a non-zero square, the two forms are isomorphic.

It follows that every quadratic form on a vector space over $\mathbb{C}$ is isomorphic to $x_1^2 + \cdots + x_n^2$, and that every quadratic form on a vector space over $\mathbb{R}$ is isomorphic to $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$ (m pluses and n minuses) for some m and n. Sylvester's law of inertia shows that these forms over $\mathbb{R}$ are non-isomorphic (proof: look at the largest possible dimension of a positive definite or negative definite subspace).

We will usually assume that N is non-degenerate (which means that the associated bilinear form is non-degenerate), but one could study Clifford algebras arising from degenerate forms. For example, the Clifford algebra of the zero form is just the exterior algebra.

Remark 190 The tensor algebra TV has a natural $\mathbb{Z}$-grading, and to form the Clifford algebra $C_V(K)$, we quotient by the ideal generated by the even elements $v^2 - N(v)$. Thus, the algebra $C_V(K) = C_V^0(K) \oplus C_V^1(K)$ is $\mathbb{Z}/2\mathbb{Z}$-graded. A $\mathbb{Z}/2\mathbb{Z}$-graded algebra is called a *superalgebra*.

We now want to solve the following problem: Find the structure of $C_{m,n}(\mathbb{R})$, the Clifford algebra over $\mathbb{R}^{n+m}$ with the form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$.

Example 191

- $C_{0,0}(\mathbb{R})$ is $\mathbb{R}$.
- $C_{1,0}(\mathbb{R})$ is $\mathbb{R}[\varepsilon]/(\varepsilon^2 - 1) = \mathbb{R}(1 + \varepsilon) \oplus \mathbb{R}(1 - \varepsilon) = \mathbb{R} \oplus \mathbb{R}$. Note that the given basis, this is a direct sum of *algebras* over $\mathbb{R}$.
- $C_{0,1}(\mathbb{R})$ is $\mathbb{R}[i]/(i^2 + 1) = \mathbb{C}$, with i odd.
- $C_{2,0}(\mathbb{R})$ is $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 - 1, \alpha\beta + \beta\alpha)$. We get a homomorphism $C_{2,0}(\mathbb{R}) \to \mathbb{M}_2(\mathbb{R})$, given by $\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\beta \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The homomorphism is onto because the two given matrices generate $\mathbb{M}_2(\mathbb{R})$ as an algebra. The dimension of $\mathbb{M}_2(\mathbb{R})$ is 4, and the dimension of $C_{2,0}(\mathbb{R})$ is at most 4 because it is spanned by 1, α, β, and $\alpha\beta$. So we have that $C_{2,0}(\mathbb{R}) \simeq \mathbb{M}_2(\mathbb{R})$.
- $C_{1,1}(\mathbb{R})$ is $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 + 1, \alpha\beta + \beta\alpha)$. Again, we get an isomorphism with $\mathbb{M}_2(\mathbb{R})$, given by $\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\beta \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$

Thus, we've computed the Clifford algebras

$m\backslash n$	0	1	2
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{M}_2(\mathbb{R})$	
2	$\mathbb{M}_2(\mathbb{R})$		