Now we will work out the roots of orthogonal groups. We first find a Cartan subgroup of $O_n(\mathbb{R})$. For example, we can take the group generated by rotations in $n/2$ or $(n-1)/2$ orthogonal planes. The problem is that this is rather a mess as it is not diagonal ("split"). So let's start again, this time using the quadratic form $x_1x_2 + x_3x_4 + \cdots + x_{2m-1} + x_{2m}$. This time a Cartan subalgebra is easier to describe: one consists of the diagaonal matrices with entries $a_1, 1/a_1, a_2, 1/a_2, \ldots$. The corresponding Cartan subalgebra consists of diagonal matrices with entries $\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots$. Now let's find the roots, in other words the eigenvalues of the adjoint representation. The Lie algebra consists of matrices A with $AJ = -JA^t$, or in other words AJ is skew symmetric, where J is the matrix of the quadratic form, which in this case has blocks of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ down the diagonal. For $m=2$ the elements of the Lie algebra are

$$\begin{pmatrix}\alpha_1 & 0 & a & b\\ 0 & -\alpha_1 & c & d\\ -d & -b & \alpha_2 & 0\\ -c & -a & 0 & -\alpha_2\end{pmatrix}$$

so the roots corresponding to the entries a, b, c, d are $\alpha_1 - \alpha_2$, $\alpha_1 + \alpha_2$, $-\alpha_1-\alpha_2$,$-\alpha_1+\alpha_2$, so are just $\pm\alpha_1 \pm \alpha_2$. (Notice that in this case the roots split into two orthogonal pairs, correspoinding to the fact that $so_4(\mathbb{C})$ is a product of two copies of $sl_2(\mathbb{C})$.) Similarly for arbitrary m the roots are $\pm\alpha_i \pm \alpha_j$. This roots system is called D_m.

Now we find the Weyl group. This consists of reflections in the roots. Such a reflection exchanges two coodinates, and possible changes the sign of both coodinates. So the Weyl group consist of linear transformations that permute the coordinates and then flip the signs of an even number of them, so has order $2^{m-1}.m!$

We will now use the root systems of orthogonal groups to explain most of the local isomorphisms and splittings we observed in low dimensions.

We first the root system to explain why orthogonal groups in 4 dimensions tend to split. Obviously the root system of a product of 2 Lie algebras is the orthogonal direct sum of their root systems. The root system of $O_4(\mathbb{C})$ happens to be a union of 2 orthogonal pairs $\pm(\alpha_1+\alpha_2)$ and $\pm(\alpha_1-\alpha_2)$, each of which is essentially the roots system of $SL_2(\mathbb{C})$, so we expect its Lie algebra to split. We should also explain why some orthogonal Lie algebras in 4 dimensions such as $o_{3,1}(\mathbb{R})$ do not split. For this, we look at the action of the Galois group of $\mathbb{C}/R$, in other words complex conjugation, on the roots. For a complex Lie algebra the roots lie in its dual. For a real lie algebra the roots need not lie in its dual: in general they are in the complexification of its dual, and in particular are acted on by complex conjugation. (For algebraic groups over more general fields such as the rationals we also get actions of the Galois group on the roots.) For the split case $o_{2,2}$ this action is trivial, for the compact group $o_{4,0}$ the action takes each root to its negative, while for $o_{3,1}$ the Galois group flips the two orthogonal pairs of roots. If the Lie algebra splits as a product then the roots system is a union of orthogonal subsystems invariant under complex conjugation, so the reason $o_{3,1}(\mathbb{R})$ does not split is that the decomposition of the root system into two orthogonal pairs cannot be done in a way invariant under complex conjugation.

Exercise 179 For orthogonal groups of odd dimensional spaces, use the quadratic form $x_0^2 + x_1x_2 + x_3x_4 + \cdots + x_{2m-1} + x_{2m}$. Show that the roots are $\pm\alpha_i \pm \alpha_j$

and $\pm\alpha_i$. This root system is called B_m. Show that the Weyl group generated by the reflections of roots has order $2^m.m!$, and is generated by permutatons and sign changes of the coordinates.

This explains why $SL_2(\mathbb{C})$ and $SO_3(\mathbb{C})$ are locally isomorphic: they have essentially the same root system consisting of two roots of sum 0.

Exercise 180 Check directly that the root systems A_3 of $SL_4(\mathbb{C})$ and D_3 of $O_6(\mathbb{C})$ are isomorphic (in other words find an isometry between the vector spaces they generate mapping the first roots system onto the second).

We will later see that $SL_5(\mathbb{C})$ and $Sp_4(\mathbb{C})$ also have the same root system, explaining why they are locally isomorphic.

Now we find the automorphism groups of these root systems B_n and D_n.

For B_m this is easy: the roots fall into m orthogonal pairs, so we can permute the m pairs and swap the elements of each pair. this gives an automorphism group $2^m.m!$, which is the same as the Weyl group.

For D_m the Weyl group is not the full automorphism group, becauase the Weyl group $2^{m-1}.m!$ can only change the sign of an even number of coordinates, but we can also swap the signs of an odd number of coordinates to get a group $2^m.m!$. In particular the root system has an automorphism of order 2 not in the Weyl group. This corresponds to an outer automorphism of the group $SO_{2m}(\mathbb{C})$, given by any determinant -1 automorphism of $O_{2m}(\mathbb{C})$.

Exercise 181 Why does this only apply for orthogonal groups in even dimensions; in other words why do the determinant -1 elements of $O_{2m+1}(\mathbb{C})$ not give outer automorphisms of $SO_{2m+1}(\mathbb{C})$?

Are there any more automorphisms? The lattice D_m (meaning the lattice generated by the roots) is a lattice of determinat 4 contained in the lattice B_m of determinant 1. If we can show that B_m is unique in some way then this will show that the automorphism group of D_m acts on B_m, so is the automorphism group of B_m. So let's look at the possible integral lattices containing D_4. Any such lattices must be contained in the dual of D_4, which consists of 4 cosets of D_4: all vecots have integral coordinates, or they all have half-integral coordinates. The B_m lattice is formed by taking D_m and adding the coset with integral coodinates whose sum is odd. So we look at the other two cosets, whose minimal norm vectors are $(\pm1/2, \pm1/2, ...,)$ with sum either even or odd. The minimal norm is $m/4$, while the minimal norm of the coset used for B_m is 1. So there are no extra automorphisms unless possibly $m/4 = 1$, in other words $m = 4$. In this case there are indeed extra automorphisms: we need to find an automorphisms acting nontrivially on these cosets, and we can find such automorphisms given by reflections in the vectors $(\pm1/2, \pm1/2, \pm1/2, \pm1/2)$ which ave norm 1 and inner product with each element of D_4 a half integer, so reflection is indeed an automorphism of D_4.