What does a general Lie group look like? In general, a Lie group G can be broken up into a number of pieces as follows.

As we mentioned earlier,the connected component of the identity, $G_{\mathrm{conn}} \subseteq G$, is a normal subgroup, and G/G_{conn} is a discrete group.

$$1 \longrightarrow G_{\mathrm{conn}} \longrightarrow G \longrightarrow G_{\mathrm{discrete}} \longrightarrow 1$$

so this breaks up a Lie group into a connected subgroup and a discrete quotient.

The maximal connected normal solvable subgroup of G_{conn} is called the solvable redical G_{sol}. Recall that a group is *solvable* if there is a chain of subgroups $G_{\mathrm{sol}} \supseteq \cdots \supseteq 1$, where consecutive Lie's theorem tells us that some cover of G_{sol} is isomorphic to a subgroup of the group of upper triangular matrices.

Since G_{sol} is solvable, $G_{\mathrm{nil}} := [G_{\mathrm{sol}}, G_{\mathrm{sol}}]$ is nilpotent, i.e. there is a chain of subgroups $G_{\mathrm{nil}} \supseteq G_1 \supseteq \cdots \supseteq G_k = 1$ such that G_i/G_{i+1} is in the center of G_{nil}/G_{i+1}. In fact, G_{nil} must be isomorphic to a subgroup of the group of upper triangular matrices with ones on the diagonal. Such a group is called *unipotent.*

Every normal solvable subgroup of $G_{\mathrm{conn}}/G_{\mathrm{sol}}$ is discrete, and therefore in the center (which is itself discrete). We call the pre-image of the center G_*. Then G/G_* is a product of simple groups (groups with no normal subgroups).

So a general Lie group has a chain of normal subgroups that are trivial,nilptent, solvable, connected, or the whole group, such that the quoteints are nilpotent, abelian, almost a product of simple groups, and discrete.

Example 11 Let G be the group of all shape-preserving transformations of $\mathbb{R}^4$ (i.e. translations, reflections, rotations, and scaling). It is sometimes called $\mathbb{R}^4 \cdot GO_4(\mathbb{R})$. The $\mathbb{R}^4$ stands for translations, the G means that you can multiply by scalars, and the O means that you can reflect and rotate. The $\mathbb{R}^4$ is a normal subgroup. In this case, we have

$$
\begin{array}{rrl}
 & \mathbb{R}^4 \cdot GO_4(\mathbb{R}) = G & \\
 & & G/G_{\mathrm{conn}} = \mathbb{Z}/2\mathbb{Z} \\
\begin{array}{c} G_{\mathrm{conn}}/G_{\mathrm{sol}} \\ = SO_4(\mathbb{R}) \end{array} & \left\{ \begin{array}{r} \mathbb{R}^4 \cdot GO_4^+(\mathbb{R}) = G_{\mathrm{conn}} \\ \\ \mathbb{R}^4 \cdot \mathbb{R}^\times = G_* \\ \\ \mathbb{R}^4 \cdot \mathbb{R}^+ = G_{\mathrm{sol}} \end{array} \right. & \begin{array}{l} G_{\mathrm{conn}}/G_* = PSO_4(\mathbb{R}) \\ \quad \big(\simeq SO_3(\mathbb{R}) \times SO_3(\mathbb{R})\big) \\ G_*/G_{\mathrm{sol}} = \mathbb{Z}/2\mathbb{Z} \end{array} \\
 & & G_{\mathrm{sol}}/G_{\mathrm{nil}} = \mathbb{R}^+ \\
 & \mathbb{R}^4 = G_{\mathrm{nil}} &
\end{array}
$$

where $GO_4^+(\mathbb{R})$ is the connected component of the identity (those transformations that preserve orientation), $\mathbb{R}^\times$ is scaling by something other than zero, and $\mathbb{R}^+$ is scaling by something positive. Note that $SO_3(\mathbb{R}) = PSO_3(\mathbb{R})$ is simple.

$SO_4(\mathbb{R})$ is "almost" the product $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$. To see this, consider the associative (but not commutative) algebra of quaternions, $\mathbb{H}$. Since $q\bar{q} = a^2 + b^2 + c^2 + d^2 > 0$ whenever $q \neq 0$, any non-zero quaternion has an inverse (namely, $\bar{q}/q\bar{q}$). Thus, $\mathbb{H}$ is a division algebra. Think of $\mathbb{H}$ as $\mathbb{R}^4$ and let S^3 be the unit sphere, consisting of the quaternions such that $\|q\| = q\bar{q} = 1$. It is easy to check that $\|pq\| = \|p\| \cdot \|q\|$, from which we get that left (right) multiplication by an element of S^3 is a norm-preserving transformation of $\mathbb{R}^4$. So we have a map $S^3 \times S^3 \to O_4(\mathbb{R})$. Since $S^3 \times S^3$ is connected, the image must lie in $SO_4(\mathbb{R})$. It is not hard to check that $SO_4(\mathbb{R})$ is the image. The kernel is $\{(1,1),(-1,-1)\}$. So we have $S^3 \times S^3/\{(1,1),(-1,-1)\} \simeq SO_4(\mathbb{R})$.

Conjugating a purely imaginary quaternion by some $q \in S^3$ yields a purely imaginary quaternion of the same norm as the original, so we have a homomorphism $S^3 \to O_3(\mathbb{R})$. Again, it is easy to check that the image is $SO_3(\mathbb{R})$ and that the kernel is ± 1, so $S^3/\{\pm 1\} \simeq SO_3(\mathbb{R})$.

So the universal cover of $SO_4(\mathbb{R})$ (a double cover) is the Cartesian square of the universal cover of $SO_3(\mathbb{R})$ (also a double cover). Orthogonal groups in dimension 4 have a strong tendency to split up like this. Orthogonal groups tend to have these double covers.

Exercise 12 Let G be the (parabolic) subgroup of $GL_4(\mathbb{R})$ of matrices whose lower left 2 by 2 block of elements are all zero. Decompose G in an analogous way to the example above.

So to classify all connected Lie groups we need to find the simple ones, the unipotent ones, and find how the simple ones can act on the unipotent ones. One might guess that the easiest part of this will be to find the unipotent ones, as these are just built from abelian ones by taking central extensions. However this turns out to be the hardest part, and there seems to be no good solution. The simple ones can be classified with some effort: we will more or less do this in the course. Over the complex numbers the complete list is given by A_n etc. (Draw diagrams). These Dynkin diagrams are pictures of the Lie groups with the following meaning. Each dot is a copy of SL_2. Two dots are disconnected if the corresponding SL_2s commute, and are joined by a single line if they "overlap by one matrix element". Double and triple lines describe more complicated ways they can interact. For each complex simple Lie group there are a finite number of simple real Lie groups whose complexification is the complex Lie group, and we will later use this to find the simple Lie groups.

For example, $\mathfrak{sl}_2(\mathbb{R}) \not\simeq \mathfrak{su}_2(\mathbb{R})$, but $\mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C} \simeq \mathfrak{su}_2(\mathbb{R}) \otimes \mathbb{C} \simeq \mathfrak{sl}_2(\mathbb{C})$. By the way, $\mathfrak{sl}_2(\mathbb{C})$ is simple as a *real* Lie algebra, but its complexification is $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, which is not simple.

Dynkin diagrams also classify lots of other things: 3-dimensional rotation groups, finite crystallographic reflection groups, du Val singularities, Macdonald polynomials, singular fibers of elliptic surfaces or elliptic curves over the integers,...

Exercise 13 Find out what all the things mentioned above are, and find some more examples of mathematical objects classified by these Dynkin diagrams. (Hint:wikipedia.)

We also need to know the actions of simple Lie groups on unipotent ones. We can at least describe the actions on abelian ones: this is called representation theory.

1.1 Infinite dimensional Lie groups

Examples of infinite dimensional Lie groups are diffeomorphisms of manifolds, or gauge groups, or infinite dimensional classical groups. There is not much general theory of infinite dimensional Lie groups:they are just too complicated.

1.2 Lie groups and finite groups

1. The classification of finite simple groups resembles the classification of connected simple Lie groups.

   For example, $PSL_n(\mathbb{R})$ is a simple Lie group, and $PSL_n(\mathbb{F}_q)$ is a finite simple group except when $n = q = 2$ or $n = 2, q = 3$. Simple finite groups form about 18 series similar to Lie groups, and 26 or 27 exceptions, called sporadic groups, which don't seem to have any analogues for Lie groups.

   Exercise 14 Show that the groups projective special linear groups $PSL_2(\mathbb{F}_4)$ and $PSL_2(\mathbb{F}_5)$ are isomorphic (or if this is too hard, show they have the same order).

2. Finite groups and Lie groups are both built up from simple and abelian groups. However, the way that finite groups are built is much more complicated than the way Lie groups are built. Finite groups can contain simple subgroups in very complicated ways; not just as direct factors.

   For example, there are *wreath products*. Let G and H be finite simple groups with an action of H on a set of n points. Then H acts on G^n by permuting the factors. We can form the semi-direct product $G^n \ltimes H$, sometimes denoted $G \wr H$. There is no analogue for (finite dimensional) Lie groups. There is an analogue for infinite dimensional Lie groups, which is one reason why the theory becomes hard in infinite dimensions.

3. The commutator subgroup of a connected solvable Lie group is nilpotent, but the commutator subgroup of a solvable finite group need not be a nilpotent group.

   Exercise 15 Show that the symmetric group S_4 is solvable but its derived subgroup is not nilpotent. Show that it cannot be represented as a group of upper triangular matrices over any field.

4 Non-trivial nilpotent finite groups are never subgroups of real upper triangular matrices (with ones on the diagonal).

1.3 Lie groups and algebraic groups

By algebraic group, we mean an algebraic variety which is also a group, such as $GL_n(\mathbb{R})$. Any real algebraic group is a Lie group. Most of the connected Lie groups we have seen so far are real algebraic groups. Since they are so similar, we'll list some differences.

1. Unipotent and semisimple abelian algebraic groups are totally different, but for Lie groups they are nearly the same. For example $\mathbb{R} \simeq \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$ is unipotent and $\mathbb{R}^\times \simeq \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right\}$ is semisimple. As Lie groups, they are closely related (nearly the same), but the Lie group homomorphism $\exp : \mathbb{R} \to \mathbb{R}^\times$ is not algebraic (polynomial), so they look quite different as algebraic groups.

2. Abelian varieties are different from affine algebraic groups. For example, consider the (projective) elliptic curve $y^2 = x^3 + x$ with its usual group operation and the group of matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with $a^2 + b^2 = 1$. Both are isomorphic to S^1 as Lie groups, but they are completely different as algebraic groups; one is projective and the other is affine.