13 Orthogonal groups

Orthogonal groups are the groups preserving a non-degenerate quadratic form on a vector space. Over the complex numbers there is essentially only one such form on a finite dimensional vector space, so we get the complex orthogonal groups $O_n(\mathbb{C})$ of complex dimension $n(n-1)/2$, whose Lie algebra is the skew symmetric matrices. Over the real numbers there are several different forms. By Sylvester's law of inertia the real nondegenerate quadratic forms are determined by their dimension and signature, so we get groups $O_{m,n}(\mathbb{R})$ preserving the form $x_1^2 + \cdots x_m^2 - x_{m+1}^2 - \cdots - x + m + n^2$. If the form is postive definite the corresponding group is compact. We have the usual variations: special orthogonal groups SO of elements of determinant 1, and projective orthogonal groups where we quotient out by the center ± 1. There are also orthogonal groups over other fields and rings corresponding to quadratic forms, whose classification over the rationals or number fields or the integers is a central part of number theory. For example, the Leech lattice Λ is a 24-dimensional quadratic form over the integers, and the corresponding orthogonal group $O_\Lambda(\mathbb{Z})$ is a double cover of the largest Conway sporadic simple group.

We first look at a few small cases.

Dimension 0 and 1 there is not much to say: theo orthogonal groups have orders 1 and 2. They are counterexamples to a surprisingly large number of published theorems whose authors forgot to exclude these cases.

Dimension 2: The special orthogonal group $SO_2(\mathbb{R})$ is the circle group S^1 and is isomorphic to the complex numbers of absolute value 1. To make things more interesting we will look at it over the rationals, in other words looks at the group $SO_2(\mathbb{Q})$. The elements of this group can be identified with Pythagorean triangles: integer solutions of $x^2 + y^2 = z^2$ with no common factor and $z > 0$ (corresponding to the point $x/z, y/z) \in SO_2(\mathbb{Q})$).

Exercise 169 Show that the points can be parametrized by $t \in \mathbb{Q} \sup \infty$ by drawing the line through $(-1, 0)$ and $(x/z, y/z)$ and taking the intersection $(1, t)$ of this with the line $(1, *)$. What is the rational number t corresponding to $3^2 + 4^2 = 5^2$? Find the Pythagorean triangle corresponding to the square of the group element corresponding to the $(3, 4, 5)$ triangle.

Exercise 170 Show that under the action of $SO_2(\mathbb{C})$, the space C^2 splits as the sum of two 1-dimensional representations. What well-known group is $SO_2(\mathbb{C})$ isomorphic to, and what are the two corresponding 1-dimensional representations of this group?

Dimension 3: We have a compact group $O_3(\mathbb{R})$, a complex group $O_3(\mathbb{C})$, and another group $O_{2,1}(\mathbb{R})$ to investigate. We say earlier that $O_{2,1}(\mathbb{R})$ is locally isomorphic to $SL_2(\mathbb{R})$.

Exercise 171 Show that $O_3(\mathbb{C})$ is locally isomorphic to $SL_2(\mathbb{C})$. (Hint: over $\mathbb{C}$ the forms $x^2 + y^2 + z^2$ and $x^2 + y^2 - z^2$ are equivalent!)

The quaternions are a useful way to describe the compact group $SO_3(\mathbb{R})$ in detail. Recall that the quaternions are a 4-dimensional division algebra over the reals with a basis $1, i, j, k$ and products $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The conjugation is defined by $\overline{a + bi + cj + dk} =$

$a-bi-cj-dk$ and the norm is defined by $N(q) = q\overline{q} = a^2+b^2+c^2+d^2$. Since this is real, any non-zero quaternion has an inverse $\overline{q}/N(q)$. The unit quaternions $q = a + bi + cj + dk$ with $N(q) = q\overline{q} = a^2 + b^2 + c^2 + d^2 = 1$ form a group homeomorphic to the sphere S^3. This is almost the same as the orthogonal group $SO_3(\mathbb{R}$. To see this consider the adjoint action $\gamma(v) = \gamma v\gamma^-1$ of the unit quaternions S^3 on R^3, identified with the space of imaginary quaternions. This preserves norms and is therefore a rotation, so we get a homomorphism $S^3 \mapsto SO_3(R)$.

Exercise 172 Show that this homomorphism is onto, and has kernel of order 2 given by $\{1, -1\}$.

This gives a fast way to multiply two rotations, since multiplying two quaternions takes fewer operations that multiplying two 3 by 3 matrices.

Exercise 173 The quaternions contain a copy of the complex numbers $a + bi$ so can be thought of as a 2-dimensional right vector space over the complex numbers. Show that under this identification, the group S^3 of unit quaternions, acting by left multiplication on $\mathbb{H} = \mathbb{C}^2$, is identified with the group SU_2.

Dimension 4. Here there are 3 real orthogonal groups $O_4(\mathbb{R})$, $O_{3,1}(\mathbb{R})$, and $O_{2,2}(\mathbb{R})$ to look at. We saw earlier that $O_{3,1}(\mathbb{R})$ is locally isomorphic to $SL_2(\mathbb{C})$, so we will mostly ignore it.

We can also use quaternions to do the next case $SO_4(R)$. To do this, be observe that both left multiplication $x \mapsto \gamma x$ and right multiplication $x \mapsto x\gamma$ by unit quaternions γ preserve the norm and therefore give rotations of 4-dimensional space (identified with the quaternions). So we get a homomorphism $S^3 \times S^3 \mapsto SO_4(\mathbb{R})$, taking $(\gamma, \delta) \in S^3 \times S^3$ to the rotation $x \mapsto \gamma x\delta^{-1}$.

Exercise 174 Check that this is a homomorphism onto $SO_4(\mathbb{R})$ with kernel $(-1, -1)$.

The group $SO_{2,2}(\mathbb{R})$, or rather a double cover of it, has a similar splitting. To see this we think of R^4 as the 2 by 2 matrices, with the determinant as a quadratic form of signature (2,2). Then left and right multiplication by elements of $SL_2(\mathbb{R})$ preserve this form, so we get a homomorphism from $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ to $O_{2,2}(\mathbb{R})$ with kernel of order 2. We get a similar splitting of the double cover of $O_4(\mathbb{C})$.

Notice that the Lie algebras of $O_4(\mathbb{R})$, $O_{3,1}(\mathbb{R})$, and $O_{2,2}(\mathbb{R})$ are all real forms of $O_4(\mathbb{C})$, but the first and third split as a product of 2 smaller Lie algebras, while the middle one is a simple Lie algebra.

Dimension 5: Orthogonal groups in dimension 5 are sometimes locally isomorphic to symplectic group; we will discuss this later when we cover symplectic groups.

Dimension 6. The orthogonal group $O_6(\mathbb{C})$ is locally isomorphic to $SL_4(\mathbb{C})$. To see this take the alternating square $\Lambda^2(\mathbb{C}^4)$. This is acted on by $SL_4(\mathbb{C})$ with kernel ± 1, and there is a symmetric bilinear form from $\Lambda^2(C^4) \times \Lambda^2(\mathbb{C}^4) \mapsto \Lambda^4(\mathbb{C}^4)$. But $\Lambda^4(\mathbb{C}^4)$ can be identified with $\mathbb{C}$ in a way that is preserved by $SL_4(\mathbb{C})$, as this identification is essentially a determinant. So we get a homomorphism from $SL_4(\mathbb{C})$ to $SO_6(\mathbb{C})$ with kernel ± 1.

Exercise 175 Similarly the group $SL_4(\mathbb{R})$ is locally isomorphic to one of the groups $SO_6(\mathbb{R})$, $SO_{5,1}$, $SO_{4,2}(\mathbb{R})$, or $SO_{3,3}(\mathbb{R})$; which?

Now we will look at some of the symmetric spaces associated to orthogonal groups, which can be thought of as the most natural things they act on. A maximal compact subgroup of $O_{m,n}(\mathbb{R})$ is $O_m(\mathbb{R}) \times O_n(\mathbb{R})$. The symmetric space is easy to identify explicitly: the orthogonal group $O_{m,n}(\mathbb{R})$ acts on $\mathbb{R}^{m+n}$, and the maximal compact subgroup $O_m(\mathbb{R}) \times O_n(\mathbb{R})$ is just the subgroup fixing the positive definite subspace $\mathbb{R}^m$. So the symmetric space is just the Grassmannian of maximal positive definite subspaces of $\mathbb{R}^{m+n}$ (an open subset of the Grasssmannian of all m-dimensional subspaces). For small values of m or n this can be described in other ways as follows.

Example 176 The orthogonal group $O_{n+1}(\mathbb{R})$ is the group of isometries of the n sphere, so the projective orthogonal group $PO_{n+1}(\mathbb{R})$ is the group of isometries of elliptic geometry (real projective space) which can be obtained from a sphere by identifying antipodal points. (Recall that P means quotient out by the center, of order 2 in this case.) We will show that the group of isometries of hyperbolic geometry can be described in a similar way.

We construct a model of hyperbolic geometry. Take the indefinite space $R^{1,n}(\mathbb{R})$ with quadratic form $x_1^2 - x_2^2 - x_3^2 - \cdots$. Then the norm 1 vectors form a 2-sheeted hyperboloid, and on this hyperboloid the pseudo-Riemannian metric of $R^{1,n}(\mathbb{R})$ restricts to a Riemannian metric. Then one of these sheets forms a model of hyperbolic space. So just as in the elliptic case, the group of isometries is $PO_{1,n}(\mathbb{R})$. In the indefinite case the orthogonal group $O_{1,n}(\mathbb{R})$ splits as a product of the center $\{1, -1\}$ of order 2 and its index 2 subgroup group $O_{1,n}(\mathbb{R})^+$ of elements that fix the two components of the hyperboloid (these are the elements of spinor norm equal to the determinant).

Exercise 177 In $\mathbb{R}^{1,n}$ there are two sorts of reflection, because we can reflect in the hyperplanes orthogonal to either positive or negative norm vectors. What two sorts of isometries of hyperbolic space do these correspond to?

The symmetric spaces of $O_{m,n}(\mathbb{R})$ for m (or n) equal to 2 also have a special property. In this case the maximal compact subgroup has a factor of $O_2(\mathbb{R})$ which is almost the circle group S^1, which strongly suggests that this symmetric space should be Hermitian. To see this we identify the symmetric space with an open subset of a complex quadric as follows. The quatric is the set of points ω of the projective space $PC^{2,n}$ such that $(\omega, \omega) = 0$. The open subset is the points with $(\omega, \overline{\omega}) > 0$. If ω is represented by $x + iy$ then $x^2 = y^2 > 0$ and $(x, y) = 0$ so x and y form an orthogonal base for a positive definite subspace of $R^{2,n}$. Changing ω by a complex scalar does not change this 2-dimensional subspace (though it changes the basis of course). So this identifies the symmetric space with an open subset of a complex variety in a natural way. These symmetric spaces turn up quite often in moduli space problems: as a typical example, the moduli space of marked Enriques surfaces is the symmetric space of $O_{2,10}(\mathbb{R})$ with a codimension 1 complex submanifold removed.

Exercise 178 Show that this symmetric space can also be identifies with the points of $\mathbb{C}^{1,n-1}$ whose imaginary part lies in the interior of one of the two cones of $\mathbb{R}^{1,n-1}$. This gives a representation of the symmetric space as a "tube domain", generalizing the upper half plane.