For a semisimple Lie group one can ask what are its "natural" actions on mathematical objects; of course this is a rather vague question. We have seen one answer above: the general linear group acts nauturally on projective spaces, Grasmannians, and flag manifolds. These are quotients by parabolic subgroups. Another class of objects acted on by semisimple Lie groups are the symmetric spaces. Symmetric spaces are generalizations of spherical and hyperbolic geometry. Their definition looks a little strange at first: they are connected Riemannian manifolds such that at each point there is an automorphism acting as $-1$ on the tangent space. Obvious examples are Euclidean space (curvature 0), spheres (positive curvature) and hyperbolic space (negative curvature), and in general symmetric spaces should be thought of as generalizations of these three fundamental geometries. (They account for many of the "natural" geometries but not for all of them: for example, 5 or the 8 Thurston geometries in 3 dimensions are symmetric spaces, but 3 of them are not.) We may as well assume that they are simply connected (by taking a universal cover) and irreducible (not the products of things of smaller dimension). These were classified by Cartan. The first major division is intor those of positive, zero, or negative curvature. The only example of zero curvature is the real line. Cartan showed that irreducible simply connected symmetric spaces of negative curvature correspond to the non-compact simple Lie groups: the symmetric space is just the Lie group modulo its maximal compact subgroup. He also showed that the ones of positive curvature correspond to the ones of negative curvature by a duality extending the duality between spherical and hyperbolic geometry. We will look at some examples of symmetric spaces related to general linear groups.

We first examine the symmetric space of the general linear group $GL_n(\mathbb{R})$ (which is neither semisimple nor simply connected, but never mind). We saw above that the maximal compact subgroup is the orthogonal group. The symmetric space $GL_n(\mathbb{R})/O_n(\mathbb{R})$ is the space of all positive definite symmetric bilinear forms on $R^n$. This is an open convex cone in $R^{n(n+1)/2}$, and is an example of a homogeneous cone. (In dimensions 3 and above it is quite rare for open convex cones to be homogeneous). A minor variation is to replace the general linear group by the special linear group, when the symmetric space becomes positive definite symmetric bilinear forms of discriminant 1.

**Exercise 158** Find a similar description of the symmetric space of $GL_n(\mathbb{C})$.

Let us look at the case $n = 2$ in more detail. In this case the space $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ can be identified with the upper half plane, as $SL_2(\mathbb{R})$ acts on this by $z \mapsto (az + b)/(cz + d)$ and $SO_2(\mathbb{R})$ is the subgroup fixing $i$. This symmetric space has a complex structure preserved by the group, so is called a Hermitian symmetric space. This means that we can construct lots of Riemann surfaces by taking the quotient of the upper half plane by a Fuchsian group (a discrete subgroup of $SL_2(\mathbb{R})$, such as $SL_2\mathbb{Z})$). In fact any Rieman surface of negative Euler characteristic can be constructed like this: this gives all of them except for the Riemann sphere with at most 2 points removed or an elliptic curve. The symmetric spaces of other real general linear or special linear groups do not have complex structures; one reason $SL_2(\mathbb{R})$ does is that $SL_2(\mathbb{R})$ happens to be $Sp_2(\mathbb{R})$, and we will see later that $Sp_{2n}(\mathbb{R})$ has a Hermitian symmetic space. We can spot plausible candidates for Hermitian symmetric spaces as follows: for a Hermitean symmetric space, we can fix some point $x$ and act on its tangent space by multiplying by complex numbers of absolute value 1. This gives a subgroup in the center of the subgroup $K$ fixing $x$ isomorphic to the circle group. This shows that we should look out for Hermitian symmetric space structures whenever the maximal compact subgroup has an $S^1$ factor.

The upper half plane is also a model of the hyperbolic plane. This does not generalize to $SL_n(\mathbb{R})$ either: the group $PSL_2(\mathbb{R})$ happens to be isomorphic to $PSO^+({}_{1,2}(\mathbb{R})$, and it is the groups $O_{1,n}$ rather than $SL_n$ that correspond to hyperbolic spaces, as we will see when we discuss orthogonal group later on.

**Exercise 159** Polar coordinates for symmetric spaces: Show that $GL_n(\mathbb{R}) = KAK$ where $K = O_n(\mathbb{R})$ is the maximal compact subgroup and $A$ is the group of positive diagonal matrices. Show that the element of $A$ in this decomposition need not be uniquely determined, but is uniquely determined up to conjugation by an element of the Weyl group. Find a geometric interpretation of the entries of $A$ in terms of the image of the unit ball under an element of $GL_n(\mathbb{R})$.

The Cartan decomposition of the Lie algebra $g = sl_n(\mathbb{R})$ is $g = k + p$, where $k$ is the subalgebra of orthogonal matrices, and $p$ is the orthogonal complement of $k$ under the Killing form, so is the subspace of symmetric matrices of trace 0. Warning:$p$ is not a subalgebra! There are the $+1$ and $-1$ eigenspaces of the Cartan involution $\theta$.

**Exercise 160** Show that the exponential map is an isomorphism from $p$ to its image $P$ in $G$, and can be identified with the symmetric space of $G$. In particular this shows that the symmetric space is contractible. For the special linear group this is easy to see directly by identifying the symmetric space with positive definite symmetric forms of determinant1, but the argument using the exponential map works for all semisimple Lie groups.

Now we look at the symmetric space of $SL_2(\mathbb{C})$. This turns out slightly surprisingly to be 3-dimensional hyperbolic space. This means that one can construct lots of hyperbolic 3-manifolds and orbifolds by taking a Kleinian group (a discrete subgroup of $SL_2(\mathbb{C})$, such as $SL_2(\mathbb{Z}[i])$, and taking the quotient of hyperbolic space by this subgroup. To see that its symmetric space is 3-dimensional hyperbolic space we first construct a homomorphism from $SL_2(\mathbb{C})$ to $O_{1,3}(\mathbb{R})$ as follows. The group $SL_2(\mathbb{C})$ acts on the space of hermitian matrices $x$ by $g(x) = gx\overline{g}^t$ and preserves the determinant of $x$. However the determinant on 2 by 2 hermitian matrices is a quadratic form of signature $(1, 3)$ so we get a homomorphism from $SL_2(\mathbb{C})$ to $O_{1,3}(\mathbb{R})$. Now the group $O_{1,3}(\mathbb{R})$ acts on the norm 1 vectors of $R^{1,3}$, which form two components, each isometric to 3-dimensional hyperbolic space. The subgroup of $O_{1,3}(\mathbb{R})$ fixing a point of hyperbolic space is a maximal compact subgroup $O_1(\mathbb{R}) \times O_3(\mathbb{R})$.

The group $O_{1,3}(\mathbb{R})$ is the Lorentz group of special relativity, and the local isomorphism with $SL_2(\mathbb{C})$ is used a lot: for example the 2-dimensional representation of $SL_2(\mathbb{C})$ and its complex conjugate are essentially the half-spin representations of a double cover of $SO_{1,3}(\mathbb{R})$. In special relativity the symmetric space appears as the possible values of the momentum of a massive particle.

(Notice by the way that whether a symmetric space has a complex structure has little to do with whether its group has one: the symmetric space of $SL_2(\mathbb{R})$ has a complex strcuture, but that of $SL_2(\mathbb{C})$ does not.)

There are several other symmetric spaces related to the general linear group. Recall that the group $GL_n(\mathbb{R})$ is not the only rela form of $GL_n(\mathbb{C})$; the group $U_n(\mathbb{R})$ of elements preserving the Hermitian form $z_1\overline{z_1}+\cdots$ . is also a real form. There is no particular reason why we sould restrict to positive definite Hermitian forms: we can also form the group $U_{m,n}(\mathbb{R})$ fixing a Hermitian form on $\mathbb{C}^{m+n}$ of signature $(m,n)$. A maximal compact subgroup of this is $U(m)\times U(n)$. Even if we restict to elements of determinant 1 this still has an $S^1$ factor, at least at the Lie algebra level, so we expect the corresponding symmetric spaces to have complex structures.

**Exercise 161** Show directly that the symmetric space of $U_{m,n}(\mathbb{R})$ is hermitian, by identifying it with the open subspace of the Grassmannian $G_{m,n}(\mathbb{C})$ consisting of the $m$-dimensional subspaces on which the Hermitian form is positive definite.

**Exercise 162** Show that the symmetric space of $U_{1,n}(\mathbb{R})$ can be identified with the unit ball in $\mathbb{C}^n$ as follows: an element of the Grassmannian represented by the point $z_0, z_1, \ldots, z_n$ is mapped to the point $(z_1/z_0, \ldots, z_n/z_0)$ of the open unit ball in $C^n$. Show that for $n=1$ this is essentially the group of Moebius transformations acting on the unit disk of the complex plane.

In particular the open unit ball in $\mathbb{C}^n$ is a bounded homogeneous domain: a bounded open subset of complex affine space such that its group of automorphisms acts transitively. (It is not quite trivial to see that there are any examples of these: the group of affine transformations of $\mathbb{C}^n$ does not act transitively on points of a bounded homogeneous domain!) In 1-dimension the Riemann mapping theorem implies that any simply connected bounded open subset of the complex plane is homogeneous, but in higher dimensions homogeneous domains are quite rare. The non-compact Hermitian symmetric spaces give examples, and for several decades it was an open problem to find any others. This was finally solved by Piatetski-Shapiro in 1959, who found an example of a 4-dimensional homogeneous bounded domain that was not a Hermitian symmetric space.

So far we have seen lots of examples of noncompact symettric spaces, constructed as $G/K$ where $K$ is a maximal compact subgroup of $G$. This construction just gives the trivial 1-point space for $G$ compact, so we need a different way of constructing compact symmetric spaces, such as spheres. Cartan discovered a duality between compact and non-compact irreducible symmetric spaces: roughly speaking, for each non-compact symmetric space there is a corresponding compact on (ignoring minor problems about abelian factors and connectedness). A well known special case of this is the duality between spherical and hyperbolic geometry (or even the duality between trigonometric functions nad hyperbolic ones). This duality works as folllows. Pick a non-compact symmetric space and look at the Cartan decomposition $k+p$ of its Lie algebra. Now change this to the Lie algebra $k+ip\subset g\otimes\mathbb{C}$. Cartan showed that this is the Lie algebra of a compact group, and the quotient of this group by the subgroup $K$ is the dual compact symmetric space. Let us find the dual comapct symmetric spaces of some of the symmetric spaces above.

**Example 163** The symmetric space of $GL_n(\mathbb{R})$ is $GL_n(\mathbb{R})/O_n(\mathbb{R})$. To find the compact dual, we first look at the Cartan decomposition $gl_n(\mathbb{R})=k+p$, where

$k$ is the Lie algebra of skew symmetric matrices, and $p$ is the space of symmetric matrices. Then we form the Lie algebra $k + ip$. This is the Lie algebra of skew Hermitian martices, so its Lie group is the unitary group. So the symmetric space should be $U(n)/O_n(\mathbb{R})$. This is the set of real forms on $C^n$ that are compatible with the Hermitian metric (in other words complex conjugation is an isometry).

**Example 164** The symmetric space of $U_{m,n}(\mathbb{R})$ is $U_{m,n}(\mathbb{R})/U(m) \times U(n)$. Changing $U_{m,n}(\mathbb{R})$ to its compact form gives $U(m+n)$. So the compact form of the symmetric space is $U(m+n)/U(m)\times U(n)$, in other words the Grassmannian $G_{m,n}(\mathbb{C})$.

**Exercise 165** Show that the compact symmetric space dual to the symmetric space of $GL_n(\mathbb{C})$ (the positive definite Hermitian forms) is the unitary group $U_n$. (Similarly all simply connected compact groups are symmetric spaces, and are the duals of the symmetric spaces of complex Lie groups.)