The diagonal matrices of $GL_n(\mathbb{R})$ form a Cartan subgroup.

**Exercise 150** Find two non-isomorphic Cartan subgroups of $GL_2(\mathbb{R})$.

We recall that a root space is an eigenspace for a non-zero eigenvalue of a Cartan subalgebra. For the general linear group the root spaces just correspond to the off-diagonal matrix entries. If $\alpha_i$ is the value of the $i$th diagonal element of a matrix, then the roots of $GL_n(\mathbb{R})$ are the linear forms $\alpha_i - \alpha_j$ for $i \neq j$. For example the roots system of $GL_2(\mathbb{R})$ is two opposite points, the roots system of $GL_3(\mathbb{R})$ is a hexagon, and the root system of $GL_4(\mathbb{R})$ is the centers of the edges of a cube. These root systems are very symmetric, and are acted on by $S_2$, $S_3$, and $S_4$. This can be ssen by identifying the symmetric groups with the permutation matrices, that normalize the diagonal matrices and therefore act on the root systems. In general the Weyl group is a quotient $N/H$ where $H$ is a Cartan subalgebra and $N$ is a group normalizing it; for the general linear group the group $H$ is the diagonal matrices, the group $N$ is the monomial matrices, and the Weyl group is the symmetric group.

**Warning 151** For the general linear group the Weyl group is a subgroup, but this is not always true; in general the Weyl group is only a subquotient. For example, for the group $SL_2(\mathbb{R})$, the Weyl group has order 2 and acts on the diagonal matrices by inversion, but $SL_2(\mathbb{R})$ has no element of order 2 acting in this way (though it does have such an element of order 4).

**Exercise 152** Show that the Lie algebra $sl_n(\mathbb{R})$ is simple for $n \geq 2$. (Show that any ideal must contain an eigenvalue of the Cartan subalgebra, then show that a non-zero element of a root space leads to a non-zero element of the Cartan subalgebra, and show that elements of the Cartan subalgebra with 2 distinct entries lead to elements of root spaces. By repeating these operations show that the ideal generated by any non-zero eigenvector is the whole Lie algebra.)

**Exercise 153** If $k$ is a field of characteristic $p$ dividing $n$, show that the Lie algebra $sl_n(k)$ is not simple. Where does the proof in the previous exercise break down? Show that the center is 1-dimensional and the quotient $psl_n(k)$ by the center is simple unless $p = n = 2$.

**Exercise 154** Most of the time, one expects an algebraic group over some field to be simple or solvable if and only if the corresponding Lie algebra has the same property. However there are exceptions to this: show that

- The group $PSL_2(\mathbb{F}_2)$ is solvable, and the Lie algebra $psl_2(\mathbb{F}_2)$ is solvable.
- The group $PSL_2(\mathbb{F}_3)$ is solvable, and the Lie algebra $psl_2(\mathbb{F}_3)$ is simple.
- The group $PSL_2(\mathbb{F}_4)$ is simple, and the Lie algebra $psl_2(\mathbb{F}_4)$ is solvable.
- The group $PSL_2(\mathbb{F}_5)$ is simple, and the Lie algebra $psl_2(\mathbb{F}_5)$ is simple.

**Exercise 155** Show that the general linear group has the Bruhat decomposition $GL_n(\mathbb{R}) = \biguplus_w BwB$ as a disjoint union of double cosets of $B$, where the union is over the $n!$ lements $w$ of the Weyl group, and $B$ is the Borel subgroup of upper triangular matrices. (If $g$ is an element of $GL_n(\mathbb{R})$, pick the first nonzero entry in the bottom row and multiply on the left and right by elements of $B$ to clear out its row and column. Then pick the first nonzero element on the next to last row, and carry on like this to get a permutation matrix.)

**Exercise 156** Show that the Bruhat decompostion induces a decomposition of the full flag manifold $G/B$ is the disjoint union of $n!$ affine spaces of various dimensions. For $GL_3$ show that these dimensions are 0, 1, 1, 2, 2, 3. Use this to calculate the cohomology groups with compact support of the space of full flags of $C^3$ if you know what this means.

**Example 157** We can calculate the number of elements of $GL_n(\mathbb{F}_q)$ as follows. It is equal to the number of bases of $\mathbb{F}_q^n$, which is just $(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$. This is fast but does not generalize in any obvious way to other finite simple groups. A more complicated way to work out the order that does generalize to all other finite groups of Lie type is to use the Bruhat decomposition. For $G = GL_n(\mathbb{F}_q)$, the order is $|G/B||B|$ where $B$ is the Borel subgroup of upper triangular matrices, that has order $(q-1)^n q^{1+2+\cdots+(n-1}$, and $G/B$ is the full flag variety. We can work out the number of points in this by using the Bruhat decomposition to decompose it into a union of $n!$ affine spaces of various dimensions. For example, for $GL_3$ the affine spaces have dimensions 0, 1, 1, 2, 2, 3, so the flag variery has order $q^0+q^1+q^1+q^2+q^2+q^3$. In general these exponents are the lengths of the elements of the symmetric group (or Weyl group) as words in the $n-1$ generators (12), (23), (34), $\ldots$ (these are not all transposition, but only the "simple" ones exchanging two adjacent numbers).