**Proof**

Take any abelian subalgebra $H$ of the Lie algebra $G$, and decompose $G$ into a direct sum of generalized eigenspace of $H$ (acting on $G$ by the adjoint representation). The eigenvalues are elements of the dual of $H$. If $G_\lambda$ is the generalized eigenspace for some eigenvalue $\lambda$, then $[G_\lambda, G_\mu] \subseteq G_{\lambda+\mu}$. In particular $G_0$ is a self-normalizing subalgebra of $G$ containing $H$. If in addition all elements of $H$ are semisimple, then $H$ lies in the center of $G_0$ as generalized eigenvectors (with eigenvalue 0) are honest eigenvectors. □

For semisimple Lie algebras we will later see that a maximal toral subalgebra is its own normalizer (so is a Cartan subalgebra). In general this is not true:

**Exercise 137** Show that a subalgebra of a nilpotent Lie algebra is toral if and only if it is contained in the center. So the center is the unique maximal toral subalgebra, and its normalizer is the whole algebra.

This exercise is a bit misleading. There is a subtle problem in that the definitino of a maximal toral subgroup of an algebraic group does not quite correspond to maximal toral subalgebras of the Lie algebra. This is because a toral subgroup of an algebraic group has elements that are semisimple. In (say) the group of unipotent upper triangular matrices, the only semisimple element is 1, so the maximal toral subgroup is trivial. However the maximal toral subalgebra of its Lie algebra is the center which is not trivial. This is related to the fact that it is ambiguous whether elements of the center of a Lie algebra or group should be thought of as semisimple or unipotent/nilpotent. For example, in the Heisenberg algebra, the center looks nilpotent in finite dimensional (algebraic) representations, but looks semisimple in its standard infinite dimensional representation. The Heisenberg algebra is trying hard to be semisimple in some sense; in fact it can be thought of as a sort of degeneration of a semisimple algebra. For semisimple lie algebras or groups this problem does not arise: "maximal toral" means the same whether one defines it algebraically or analytically.

**Exercise 138** Show that $sl_2(R)$ has two maximal toral subalgebras that are not conjugate under any automorphism. (Take one to correspond to diagonal matrices, and the other to correspond to a compact group of rotations.)

**Theorem 139** *If a finite dimensional complex Lie algebra is semisimple, then the normalizers of the maximal toral subalgebras are abelian*

**Proof** Suppose $H$ is a maximal toral subalgebra, and $G_0$ its normalizer, so that $G_0$ is nilpotent. Since $G_0$ is solvable it can be put into upper triangular form, so the Killing form restricted to $G_0$ has $[G_0, G_0]$ in its kernel. On the other hand, any invariant bilinear form vanishes on $(u, v)$ if $u$ and $v$ have eigenvalues that do not sum to 0, so $G_0$ is orrthogonal to all other eigenspaces of $H$. So $[G_0, G_0]$ is in the kernel of the Killing form. By Cartan's criterion, this implies that it vanishes, so $G_0$ is abelian. □

**Remark 140** There is an analogue of Cartan subgroups for finite groups. A subgroup of a finite group is called a Carter subgroup (not a misprint: these are named after Roger Carter) if it is nilpotent and self-normalizing. Any solvable finite group contains Carter subgroups, and any two Carter subgroups of a finite group are conjugate. However anyone with plans to classify the finite simple groups by copying the use of Cartan subgroups in the classification of simple Lie groups should take note of the following exercise:

**Exercise 141** Show that the simple group $A_5$ of order 60 does not have any Carter subgroups.

The analogues of Cartan subgroups for compact Lie groups are maximal tori. In fact these are the subgroups associated to Cartan subalgebras of the Lie algebra. Every element of a compact connected Lie group is contained in a maximal torus, and the maximal tori are all conjugate.

**Warning 142** In a compact connected Lie group, maximal tori are maximal abelian subgroups, but the converse is false in general: maximal abelian subgroups of a compact connected Lie group are not necessarily maximal tori. This is a common mistake. In particular, although every element is contained in a torus, it need not be true that every abelian subgroup is contained in a torus.

**Exercise 143** Show that the subgroup of diagonal matrices of $SO_n(\mathbb{R})$ for $n \geq 3$ is a maximal abelian subgroup but is not contained in any torus.

# 12 Unitary and general linear groups

The fundamental example of a Lie group is the general linear group $GL_n(\mathbb{R})$. There are several closely related variations of this:

- The complex general linear group $GL_n(\mathbb{C})$
- The unitary group $U_n$
- The special linear groups or special unitary groups, where one restricts to matrices of determinant 1
- The projective linear groups, where one quotients out by the center (diagonal matrices)

**Exercise 144** Show that the complex Lie algebras $gl_n(\mathbb{R}) \otimes \mathbb{C}$, $gl_n(\mathbb{C})$, and $u_n(\mathbb{R}) \otimes \mathbb{C}$ are all isomorphic. We say that $gl_n(\mathbb{R})$ and $u_n(\mathbb{R})$ are real forms of $gl_n(\mathbb{C})$.

The general linear group has a rather obvious representation on $n$-dimensional space. Therefore it also acts on the 1-dimensional subspaces of this, in other words $n-1$-dimensional projective space. The center acts trivially, so we get an action of the projective general linear group $PGL_n(\mathbb{R})$ on $P^{n-1}$. There is nothing special about 1-dimensional subspaces: the general linear group also acts on the Grassmannian $G(m, n-m)$ of $m$-dimensional subspaces of $\mathbb{R}^n$. The subgroup fixing one such subspace is the subgroup of block matrices $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, so the Grassmannian is a quotient of these two group.

**Exercise 145** Show that the Grassmannian is compact. (This also follows from the Iwasawa decomposition below).

More generally still, we can let the general linear group act on the flag manifolds, consisting of chains of subspaces $0 \subset V_1 \subset V_2 \cdots$, where the subspaces have given dimensions. The extreme case is when $V_i$ has dimension $i$, in which case the subgroup fixing a flag is the Borel subgroup of upper triangular matrices. In general the subgroups fixing flags are called parabolic subgroups; the orresponding quotient spaces are projective varieties.

The Iwasawa decomposition for the general linear group is $G = GL_n(\mathbb{R}) = KAN$ where $K = O_n(\mathbb{R})$ is a maximal compact subgroup, $A$ is the abelian subgroup of diagonal matrices with positive coefficients, and $N$ is the unipotent subgroup of unipotent upper triangular matrices. For the general linear group, the Iwasawa decomposition is essentially the same as the Gram-Schmidt process for turning a base into an orthonormal base. This works as follows. Pick any base $a_1, a_2, \ldots$ of $R^n$; this is more or less equivalent to picking an an element of the general linear group. Now we can make the base orthogonal by adding a linear combination of $a_1$ to $a_2$, then adding a linear comnination of $a_1, a_2$ to $a_3$, and so on. This operation corresponds to multiplying the base by an element $N$ of the unipotent upper triangular matrices. Next we can make the elements of the base have norm 1 by multiplying them by positive reals. This corresponds to acting on the base by an element of the subgroup $A$ of diagonal matrices with positive entries. We end up with an orthonormal base, that corresponds to an element of the orthogonal group.

**Exercise 146** Show that $GL_n(\mathbb{R})$ is homeomorphic as a topological space to $K \times A \times N$ and deduce that it has the same homotopy type as the orthogonal group. Show that $GL_3(\mathbb{R})$ has a fundamental group of order 2. (The corresponding simply connected group is not algebraic.)

**Exercise 147** Show that the average of any positive definite inner product on $\mathbb{R}^n$ under a compact subgroup of $GL_n(\mathbb{R})$ is invariant under the compact subgroup. Deduce the the maximal compact subgroups of $GL_n(\mathbb{R})$ are exactly the subgroups conjuugate to the orthogonal group. (A similar statement is true for all semisimple Lie groups: the maximal compact subgroups are all conjugate.)

**Exercise 148** Show that $GL_n(\mathbb{C}) = KAN$ where $K$ is the unitary group, $A$ is the positive diagonal matrices, and $N$ is the upper triangular complex unipotent matrices. Show that $SL_n(\mathbb{C})$ has a similar decomposition with $K$ the special unitary group. Show that $SL_2(\mathbb{C})$ has the same homotopy type as a 3-sphere.

**Exercise 149** Show that the unitary group acts trasitively on the full flag manifold of $\mathbb{C}^n$. What is the subgroup of the unitary group fixing a full flag?