Exercise 125 Show that for the Lie algebra $gl_n(k)$ with e_i the matriix with just one non-zero entry, a 1 in position i on the diagonal, we have $(e_i, e_j) = 2n - 2$ if $i = j$ and -2 if $i \neq j$. Deduce that the Killing form on $sl_n(k)$ is 2n times the symmetric bilinear form associated to the standard representation, but the Killing form on $gl_n(\mathbb{R})$ is not a multiple of the form of the standard representation and has a non-trivial kernel.

Exercise 126 If k is a field of characteristic 2 then the semidirect product $gl_2(k).k^2$ is solvable. The Killing form is not identically zero on its derived algebra $sl_2(k).k^2$.

Exercise 127 Let G be the Lie algebra $sl_p(F_p)$ of 2 by 2 matrices over the field of p elements. Show that G is simple if $p > 2$. Show that the Killing form of G is identically 0. Show that Trace(AB) is a non-degenerate invariant bilinear form on G.

Exercise 128 Suppose G is the Lie algebra over a field of characteristic $p > 0$ with basis a_i for $i \in Z/pZ$ and bracket $[a_i, a_j] = (i - j)a_{i+j}$. Show that G is a simple Lie algebra but has no non-zero invariant bilinear form.

Lemma 129 *(Dieudonne) Suppose that a finite dimensional Lie algebra over any field of any characteristic has a non-degenerate bilinear form, and no abelian ideals. Then it is a direct sum of simple subalgebras.*

Proof Fix a minimal ideal M. The derived ideal $[M, M]$ is contained in M and cannot be 0 as M is non-abelian, so $M = [M, M]$ is perfect. The orthogonal complement N of M is also an ideal as the bilinear form is invariant. It cannot contain M as otherwise we would have $(x, m) = (x, \sum[a_i, b_i]) = \sum([x, a_i], b_i) = 0$ so M would be in the kernel of $(,)$ which is not possible. So $N \cap M = 0$ as it is a proper ideal of the minimal ideal M. So G splits as the direct sum of M and N, so M is simple, and continuing by induction so is N. □

Exercise 130 Show that if L is a Lie algebra with an invariant symmetric bilinear form $(,)$ then $L[t]/(t^n)$ has an invariant symmetric bilinear form given by the coefficient of t^{n-1} of the bilinear form on $L[t]/(t^n)$ with values that are truncated power series. If the form on L is non-degenerate show that the form on $L[t]/(t^n)$ is also non-degenerate. Find an example of a finite-dimensional complex Lie algebra with a non-degenerate symmetric bilinear form that is not a sum of abelian and simple Lie algebras.

Exercise 131 What is wrong with the following "proof" of the false result that a finite-dimensional complex Lie algebra L with a non-degenerate symmetric bilinear form is a sum of abelian and simple Lie algebras: take any ideal of L, and write L as the sum of the ideal and its orthogonal complement (which is also an ideal). By repeating this we can write L as a sum of ideals with no proper subideals, so L is a sum of abelian and simple Lie algebras.

Corollary 132 *(Cartan's criterion for semisimplicity) For a complex Lie algebra G the following conditions are equivalent:*

1. *G has no nonzero solvable ideals*
2. *G has no non-zero abelian ideals*
3. *G has non-degenerate Killing form*
4. *G is a direct sum of simple Lie algebras (in other words G is semisimple)*

Proof If the Killing form is degenerate, then its kernel is an ideal, and is solvable by one form of Cartan's criterion. So if the algebra has no nonzero solvable ideals then the Killing form is non-degenerate. Conversely if the Killing form is nondegenerate and A is an abelian ideal, then for any $a \in A$ and $g \in G$, $Ad(a)Ad(g)$ has square 0 so has trace 0 and therefore $a = 0$ as the Killing form is non-degenerate. So all Abelian ideals are 0,and therefore all solvable ideals are 0.

We have seen that if the Killing form is non-degenerate then there are no abelian ideals, so if the Killing form is non-degenerate then by the previous lemma the Lie algebra is a sum of simple Lie algebras.

If the algebra is a sum of simple subalgebras, it is obvious that it has no nonzero solvable ideals.

□

11 Cartan subalgebras, Cartan subgroups and maximal tori

The Lie algebra gl_n has a subalgebra H of diagonal matrices, and under the action of this subalgebra gl_n splits as the sum of eigenspaces. The zero eigenspace is just H, while the other eigenspaces just correspond to the off-diagonal entries of gl_n. The subalgebra H is an example of a Cartan subalgebra, and we want to find a similar subalgebra for any Lie algebra. The first guess is to take a maximal abelian subalgebra, but this does not work: the algebra of matrices of block form $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ is abelian but does not act nicely on the rest of the Lie algebra (and has dimension much larger than that of the diagonal matrices).

Definition 133 *A toral subalgebra of a Lie algebra is an abelian subalgebra that acts semisimply on the adjoint representation.*

Definition 134 *A Cartan of a Lie algebra is a self-centralizing nilpotent subalgebra. (Self-centralizing means that it contains its centralizer.)*

For semisimple Lie algebras, maximal toral subalgebras and Cartan subalgebras will turn out to be the same. In general it is really the maximal toral subalgebras that are important. It seems to be a historical accident that Cartan subalgebras have this rather unintuitive definition. The properties of being nilpotent or self normalizing are not really that important or easy to use. The really important property is the semisimplicity, which means that one can decompose the (complex) Lie algebra into eigenspaces.

Exercise 135 Find maximal toral subalgebras for the algebra of all matrices, the algebra of upper triangular matrices, and the algebra of strictly upper triangular matrices (0's on the diagonal).

Theorem 136 *The centralizer of a maximal toral subalgebra of G is a Cartan subalgebra, in other words it is self normalizing and nilpotent.*