10 Killing form and Cartan's criterion

At first sight one might guess that solvable groups are easier to classify than simple ones, and 0-dimensional compact simple groups are easier to classify than ones of higher dimension. This turns out to be completely wrong: the 0dimensional compact simple groups are far harder to classify than the ones of positive dimension, and the solvable ones seem hopelessly complicated. What is the key reason why the positive dimensional simple Lie groups are so much easier to handle? One answer is Cartan's criterion, which implies that the Killing form on a simple complex Lie algebra (a symmetric invariant bilinear form) is non-degenerate.

We first figure out what it means for a bilinear form (,) on a representation V of a Lie algebra to be invariant. For a group G acting on V, invariance obviously means that $(gu, gv) = (u, v)$. For a Lie algebra, we formally replace g by $1+\epsilon a$ for some a in the Lie algebra (with $\epsilon^2 = 0$), to find that invaraince means $(u, v)+\epsilon([a, u], v)+(u, [a, v]) = (u, v)$, or in other words $([a, u], v)+(u, [a, v]) = 0$.

If the Lie algebra G acts on a finite dimensional vector space V, we can define a bilinear form on G by $(a, b)_V = \mathrm{Trace}_V(ab)$.

Exercise 108 Show that this form is invariant, which means $([a, b], c)_V+(b, [a, c])_V = 0$.

A particularly important special case is where we take V to be the adjoint representation of G. In this case the invariant bilinear form on G is called the Killing form $(a, b) = \mathrm{Trace}(Ad(a)Ad(b))$.

Example 109 The Killing form on any abelian Lie algebra is obviously just zero. More generally, the Killing form on any nilpotent Lie algebra is identically zero, as we can put the matrices representing it into strictly upper triangular form, and the product of any two such matrices has trace 0. This does not mean that these algebras cannot have non-zero invariant bilinear forms; for example, any bilinear form on an abelian Lie algebra is invariant.

Exercise 110 Show that the kernel of an invariant symmetric bilinear form on a Lie algebra is an ideal. In particular if the Lie algebra is simple then the bilinear form is either zero or non-degenerate. Show that the orthogonal complement of an ideal is an ideal.

Exercise 111 Find the Killing form on the 2-dimensional non-abelian Lie algebra, and check that it is degenerate but not identically zero.

Exercise 112 Find the Killing form on the Lie algebra $su(2)$, and check that it is negative definite.

Exercise 113 Find the Killing form on the Lie algebra $sl_2(\mathbb{R})$, and check that it is non-degenerate and indefinite.

Exercise 114 If L is the complex Lie algebra spanned by W, X, Y, Z with relations $[X, Y] = Z, [W, X] = X, [W, Y] = -Y, Z \in$ center, find a non-degenerate invariant symmetric bilinear form on L. Show that the bilinear form associated to any finite-dimensional representation of L is degenerate (use Lie's theorem to put L in upper triangular form).

Theorem 115 *(Jordan decomposition) Suppose that a is a linear transformation of a vector space V over a perfect field k. Then there is a unique way to write $a = a_s + a_n$ where a_s is semisimple, a_n is nilpotent, and a_s and a_n commute.*

Proof We can assume k is algebraically closed, as uniqueness of the decomposition implies it is fixed by all elements of the absolute Galois group of k, and therefore in k as k is perfect.

For existence, write V as a direct sum of the generalized eigenspaces V_λ of a with eigenvalues λ. (Recall that v is a generalized eigenvector for eigenvalue λ if $(a-\lambda)^n v = 0$ for some positive integer n.) Then we just put $a_s = \lambda$ on V_λ, $a_n = a - a_s$ and it is easy to check that a_n and a_s have the required properties.

Uniqueness is left as an exercise. □

Exercise 116 (Jordan decomposition can fail over non-perfect fields) Suppose that V is the field $F_p(x)$ and k the subfield $F_p(x^p)$. Show that V is a vector space over k of dimension p, and multiplication by x is a linear transformation of V that cannot be written as the sum $x_s + x_n$ of commuting semisimple and nilpotent endomorphisms.

Exercise 117 (Multiplicative Jordan decomposition) Show that an invertible linear transformation a acting on a finite dimensional vector space over a perfect field can be written uniquely as $a = a_s a_u$ where a_s is semisimple and a_u is unipotent (all eigenvalues 1) and $a_s a_u = a_u a_s$.

Lemma 118 *Suppose that $M \subseteq gl(V)$ is the normalizer of a subspace G of $gl(V)$, for a complex vector space V. If a is in the kernel of the form $(,)_V$ on $M \times M$, then a is nilpotent.*

Proof The semisimple part a_s of a also lies in M because $Ad(a_s) = Ad(a)_s$. If b is in M and $Ad(c)$ is a polynomial in $Ad(b)$ then c is also in M.

Suppose that a has eigenvalues α_i. Suppose that ϕ is any additive (possibly not $\mathbb{R}$-linear) function from the rational vector space spanned by the α_i. The eigenvalues of $Ad(a)$ are $\alpha_i - \alpha_j$, so there is a polynomial p such that $p(\alpha_i - \alpha_j) = \phi(\alpha_i) - \phi(\alpha_j)$, as whenever two of the terms $\alpha_i - \alpha_j$ are equal, so are the corresponding terms $\phi(\alpha_i) - \phi(\alpha_j)$ by linearity of ϕ. So the element c with eigenvalues $\phi(\alpha_i)$ is in M because $Ad(c)$ has eigenvalues $\phi(\alpha_i) - \phi(\alpha_j)$ so is $p(Ad(a)$. But then $\sum \alpha_i \phi(\alpha_i) = (a, c) = \phi(0) = 0$. Taking ϕ to be complex conjugation shows that $\sum |\alpha_i|^2 = 0$, so all the α_i are zero. So $a_s = 0$ and a is nilpotent. □

A little more effort show that the same result holds over fields of characteristic 0, but we will not use or prove this.

Theorem 119 *Cartan's criterion for a faithful representation. Suppose that G is a subalgebra of $gl(V)$ with $(a, b)_V = 0$ for all $a, b \in G$, where V is a finite dimensional complex vector space. Then G is solvable.*

Proof Let M be the normalizer of G. Then $(m, [g_1, g_2]) = ([m, g_1], g_2) = 0$ for $g_i \in G$, so $(M, [G, G]) = 0$. By the previous lemma this implies that all elements of $[G, G]$ are nilpotent. Engel's theorem then implies that $[G, G]$ is nilpotent, so G is solvable.

□

Exercise 120 Suppose that G is a subalgebra of $gl(V)$ where V is a finite dimensional complex vector space. Show that G is solvable if and only if $(a, b)_V = 0$ for all $a, b \in [G, G]$, (Use Cartan's criterion above and Lie's theorem.)

Theorem 121 *Cartan's criterion for the Killing form. If G is a finite dimensional Lie algebra over a field of characteristic 0 whose Killing form is 0, then G is solvable.*

Proof We apply Cartan's criterion for the adjoint representation. There is a slight glitch because G need not act faithfully on the adjoint representation because its center acts trivially on G. However this is not a big deal, because we find that G/center is solvable, which immediately implies that G is also solvable.

□

Cartan's criterion as stated above does not give a necessary and sufficient condition for a Lie algebra to be solvable, because the Killing form on a solvable Lie algebra need not be zero. It is often stated as the following variation, which does give a necessary and sufficient condition.

Exercise 122 Cartan's criterion, necessary and sufficient form. If G is a finite dimensional Lie algebra over a field of characteristic 0, then the following conditions are equivalent:

- G is solvable
- $(a, b) = 0$ if a is in $[G, G]$. In other words $[G, G]$ is in the kernel of the Killing form.
- $(a, b) = 0$ if a and b are both in $[G, G]$.

(For one implication use Lie's theorem, and for another use Cartan's criterion and the fact that if $[G, G]$ is solvable then so is G.

Example 123 A nilpotent Lie algebra has a Killing form that is identically zero. The converse is not true. Suppose that V is a finite dimensional vector space with and automorphism A, and we take the Lie algebra G that is a semidirect product of V with a 1-dimensional Lie algebra whose action on V is given by A. Then G is nilpotent if and only if A is a nilpotent endomorphism. The Killing form contains V in its kernel, so vanishes on $G \times G$ if $(A, A) = 0$. But $(A, A) = \mathrm{Trace}(A^2)$, so if we take A to be any endomorphism whose square has trace 0 but is not nilpotent we get a non-nilpotent Lie algebra whose Killing form vanishes.

Exercise 124 Find a non-nilpotent 2 by 2 real matrix A whose square has trace 0.